DIRECTORATE OF DISTANCE EDUCATION

UNIVERSITY OF NORTH BENGAL

MASTER OF SCIENCES- MATHEMATICS

SEMESTER -III

FUNCTIONAL ANALYSIS

DEMATH3CORE2

BLOCK-1

UNIVERSITY OF NORTH BENGAL

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FOREWORD

The Self Learning Material (SLM) is written with the aim of providing simple and organized study content to all the learners. The SLMs are prepared on the framework of being mutually cohesive, internally consistent and structured as per the university's syllabi. It is a humble attempt to give glimpses of the various approaches and dimensions to the topic of study and to kindle the learner's interest to the subject

We have tried to put together information from various sources into this book that has been written in an engaging style with interesting and relevant examples. It introduces you to the insights of subject concepts and theories and presents them in a way that is easy to understand and comprehend.

We always believe in continuous improvement and would periodically update the content in the very interest of the learners. It may be added that despite enormous efforts and coordination, there is every possibility for some omission or inadequacy in few areas or topics, which would definitely be rectified in future.

We hope you enjoy learning from this book and the experience truly enrich your learning and help you to advance in your career and future endeavours.

FUNCTIONAL ANALYSIS

BLOCK-1

BLOCK-2

BLOCK-1 FUNCTIONAL ANALYSIS

First, we use Zorn's lemma to prove there is always a basis for any vector space. It fills up a gap in elementary linear algebra where the proof was only given for finite dimensional vector spaces. The inadequacy of this notion of basis for infinite dimensional spaces motivates the introduction of analysis to the study of function spaces. Second, we discuss three basic inequalities, namely, Young's, Holder's, and Murkowski's inequalities. We establish Young's inequality by elementary means, use it to deduce Holder's inequality, and in term use Holder's inequality to prove Murkowski's inequality.

The fundamental Hahn-Banach theorem guarantees there are sufficiently many such functional for various purposes.

The uniform boundedness principle and the open mapping theorem. Together with Hahn-Banach theorem, they form the cornerstone of the subject. Nevertheless, unlike the Hahn-Banach theorem, both theorems depend critically on completeness. Being the infinite dimensional counterpart of the eigenvalues of a matrix, spectra play an important role in analyzing bounded linear operators.

In the next two sections we shall discuss the uniform boundedness principle and the open mapping theorem both due to Banach. The underlying idea of the proofs of these theorems is the Baire theorem for complete metric spaces.

UNIT 1: NORMED SPACES I

STRUCTURE

1.0 OBJECTIVE

Understand the concept of Vector Spaces Of Functions Comprehend the concept of Zorn's Lemma, Enumerate the Existence of Basis & Normed Vector Spaces, Understand the Three Inequalities

1.1 INTRODUCTION

Generally speaking, in functional analysis we study infinite dimensional vector spaces of functions and the linear operators between them by analytic methods. This chapter is of preparatory nature. First, we use Zorn's lemma to prove there is always a basis for any vector space. It fills up a gap in elementary linear algebra where the proof was only given for finite dimensional vector spaces. The inadequacy of this notion of basis for infinite dimensional spaces motivates the introduction of analysis to the study of function spaces. Second, we discuss three basic

inequalities, namely, Young's, H¨older's, and Minkowski's inequalities. We establish Young's inequality by elementary means, use it to deduce H¨older's inequality, and in term use H¨older's inequality to prove Minkowski's inequality. The latter will be used to introduce norms on some common vector spaces. As you will see, these spaces form our principal examples
throughout this book.

1.2 VECTOR SPACES OF FUNCTIONS

Recall that a vector space is over a field 𝔽. Throughout this book it is always assumed this field is either the real field ℝ or the complex field ℂ. In the following 𝔽 stands for ℝ or ℂ. It is true that many vector spaces can be viewed as vector spaces of functions. To describe this unified point of view, let S be a non-empty set and denote the collection of all functions from S to F by $F(S)$. It is routine to check that $F(S)$ forms a vector space over F under the obvious rules of addition and scalar multiplication for functions: For $f, g \in F(S)$ and $\alpha \in \mathbb{F}$,

$$(f + g)(p) \equiv f(p) + g(p), \ (\alpha f)(p) \equiv \alpha f(p).$$

In fact, these algebraic operations are inherited from the target 𝔽.
First, take $S = \{p1, \cdots, pn\}$ a set consisting of n many elements. Every function $f \in F(S)$ is uniquely determined by its values at $p1, \cdots, pn$, so f can be identified with the n-triple $(f(p1), \cdots, f(pn))$. It is easy to see that $F(\{p1, \cdots, pn\})$ is linearly isomorphic to $\mathbb{F}^n$. More precisely, the mapping $f \mapsto (f(p1), \cdots, f(pn))$ is a linear bijection between $F(\{p1, \cdots, pn\})$ and $\mathbb{F}^n$.

Second, take $S = \{p1, p2, \cdots\}$. As above, any $f \in F(S)$ can be identified with the sequence $(f(p1), f(p2), f(p3) \cdots)$. The vector space $F(\{p_j\}_{j=1}^{\infty})$ may be called the space of sequences over 𝔽.

Finally, taking $S = [0, 1]$, $F([0, 1])$ consists of all 𝔽-valued functions. The vector spaces we are going to encounter are mostly these spaces and their subspaces.

1.3 ZORN'S LEMMA

In linear algebra, it was pointed out that every vector space has a basis no matter it is of finite or infinite dimension, but the proof was only given in the finite dimensional case. Here we provide a proof of the general case. The proof depends critically on Zorn's lemma, an assertion equivalent to the axiom of choice.

To formulate Zorn's lemma, we need to consider a partial order on a set. A relation $\leq$ on a non-empty set X is called a **partial order** on X if it satisfies

(PO1) $x \leq x, \ \forall x \in X$;

(PO2) $x \leq y$ and $y \leq x$ implies $x = y$.

(PO3) $x \leq y, \ y \leq z$ implies $x \leq z$.

The pair $(X, \leq)$ is called a partially ordered set or a poset for short. A non-empty subset Y of X is called a chain or a totally ordered set if for any two $y1, y2 \in Y$, either $y1 \leq y2$ or $y2 \leq y1$ holds. In other words, every pair of elements in Y are related. An upper bound of a non-empty subset Y of X is an element u, which may or may not be in Y, such that $y \leq u$ for all $y \in Y$. Finally, a maximal element of $(X, \leq)$ is an element z in X such that $z \leq x$ implies $z = x$.

Example. Let *Us* be a set and consider $X = P(S)$, the power set of S. It is clear that the relation "set inclusion" $A \subset B$ is a partial order on $P(S)$. It has a unique maximal element given by S itself.

Example. Let $X = \mathbb{R}^2$ and define $x \prec y$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$. For instance, $(-1, 5) \prec (0, 8)$ but $(-2, 3)$ and $(35, -1)$ are unrelated. Then $(X, \prec)$ forms a poset without any maximal element.

1.3.1 Zorn's Lemma. Let $(X, \leq)$ be a poset. If every chain in X has an upper bound, then X has at least one maximal element.

Although called a lemma by historical reason, Zorn's lemma, a constituent in the Zermelo raenkel set theory, is an axiom in nature. It is equivalent to the axiom of choice as well as the Hausdorff maximalist principle. You may look up Hewitt-Stromberg's "Real and Abstract Analysis" for further information.

1.4 EXISTENCE OF BASIS

As a standard application of Zorn's lemma, we show there is a basis in any vector space. To refresh your memory, let's recall that a subset S in a vector space X is called a linearly independent set if any finite number of vectors in S are linearly independent. In other words, letting $\{x1, \cdots, xn\}$ be any subset of S, if $\alpha 1x1 + \cdots + \alpha nxn = 0$ for some scalars αi, $i = 1, \cdots, n$, then $\alpha i = 0$ for all i. On the other hand, given any subset S, denote all linear combinations of vectors from S by$\langle S \rangle$. It is easy to check that *hSi* forms a subspace of X called the subspace spanned by S. A subset S is called a spanning set of X if $\langle S \rangle$ is X, and it is called a basis of X if it is also a linearly independent spanning set. When X admits a finite spanning set, it has a basis consisting of finitely many vectors. Moreover, all bases have the same number of vectors and we all this number the dimension of the space X. The space X is of infinite dimension if it does not have a finite spanning set.

Theorem 14.1. *Every non-zero vector space has a basis.* This basis is sometimes called a **Hamel basis**.

Proof. Let X be the set of all linearly independent subsets of a given vector space V . Since V is non-zero, X is a non-empty set. Clearly the set inclusion ⊂ makes it into a poset. To apply Zorn's lemma, let's verify that every chain in it has an upper bound. Let Y be a chain in X , consider the following subset of V

$$S = \bigcup_{C \in \mathcal{Y}} C.$$

We claim that (i) $S \in X$, that's, S is a linearly independent set, (ii) $C \subset S$, $\forall C \in Y$, that's, S is an upper bound of Y. Since (ii) is obvious, it is sufficient to verify (i).

To this end, pick $v1, \cdots, vn \in S$. By definition, we can find $C1, \cdots, Cn$ in Y such that $v1 \in C1, \cdots, vn \in Cn$. As Y is a chain, $C1, \cdots, Cn$ satisfy $Ci \subset Cj$ or $Cj \subset Ci$ for any i, j. After re-arranging the indices, one may assume $C1 \subset C2 \subset \cdots \subset Cn$, and so $\{v1, \cdots, vn\} \subset Cn$. Since C_n is a linearly independent set, $\{v1, \cdots, vn\}$ is linearly independent. This shows that S is a linearly independent set.

After showing that every chain in X has an upper bound, we appeal to Zorn's lemma to conclude

that X has a maximal element B. We claim that B is a basis for V. For, first of all, B belonging to X means that B is a linearly independent set. To show that it spans V, we pick $v \in V$. Suppose v does not belong to $\langle B \rangle$, so v is independent from all vectors in B. But then the set B e = $B \cup \{v\}$ is a linearly independent set which contains B as its proper subset, contradicting the maximalist of B. We conclude that $\langle B \rangle = V$, so B forms a basis of V.

Example. Consider the power set of $\mathbb{R}^3$ which is partially ordered by set inclusion. Let X be the

subset of all linearly independent sets in $\mathbb{R}^3$. Then

$\mathcal{Y}_1 \equiv \{\{(1, 0, 0)\}, \{(1, 0, 0), (1, 1, 0)\}, \{(1, 0, 0), (1, 1, 0), (0, 0, -3)\}\}$

and

$\mathcal{Y}_2 \equiv \{\{(1, 3, 5), (2, 4, 6)\}, \{(1, 3, 5), (2, 4, 6), (1, 0, 0)\}\}$ are chains but

$\mathcal{Y}_3 \equiv \{\{\{(1, 0, 0)\}, \{(1, 0, 0), (0, 1, 0)\}, \{(1, 0, 0), (0, -2, 0), (0, 0, 1)\}\}$ is not a chain in X.

For a finite dimensional vector space, it is relatively easy to find an explicit basis, and bases are used in many occasions such as in the determination of the dimension of the vector space and in the representation of a linear operator as a matrix. However, in contrast, the existence of a basis in infinite dimensional space is proved via a non-constructive argument. It is not easy to write down a basis.

For example, consider the space of sequences $S \equiv \{x = (x1, x2, \cdots, xn \cdots) : xi \in \mathbb{F}\}$. Letting $ej = (0, \cdots, 1, \cdots)$ where "1" appears in the j-th place, it is tempting from the formula $x = \sum_{j=0}^{\infty} xjej$ to assert that $\{e_j\}_1^{\infty}$ forms a basis for S. But, this is not true. Why?

It is because infinite sums are not linear combinations. Indeed, one cannot talk about infinite sums in a vector space as there is no means to measure convergence.

According to Theorem 1.3.1, however, there is a rather mysterious basis. In general, a non-explicit basis is difficult to work with, and thus lessens its importance in the study of infinite dimensional spaces.
To proceed further, analytical structures will be added to vector spaces. Later, we will see that for a reasonably nice infinite dimensional vector space, any basis must consist of uncountable many vectors Suitable generalizations of this notion are needed. For an infinite dimensional normed space, one may introduce the so-called Schauder basis as a replacement. For a complete inner product spaces (a Hilbert space), an even more useful notion, a complete orthonormal set, will be much more useful.
Mathematics is a deductive science. A limited number of axioms is needed to build up the tower of mathematics, and Zorn's lemma is one of them.

CHECK YOUR PROGRESS

3.What is Vector space of a function

2. Explain Existence of Basis

1.5 THREE INEQUALITIES

Now we come to Young's, Holder's and Murkowski's inequalities. Two positive numbers p and q are **conjugate** if $1/p + 1/q = 1$. Notice that they must be greater than one and q approaches infinity as p approaches 1. In the following paragraphs q is always conjugate top.

Proposition 1.5.1 (Young's Inequality). *For any* $a, b > 0$ *and* $p > 1$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

and equality holds if and only if $a^p = b^q$.

Proof. Consider the function

$$\varphi(x) = \frac{x^p}{p} + \frac{1}{q} - x, \quad x \in (0, \infty).$$

From the sign of $\phi'(x) = x^{p-1} - 1$ we see that ϕ is strictly decreasing on $(0, 1)$ and strictly increasing on $(1, \infty)$. It follows that $x = 1$ is the strict minimum of ϕ on $(0, \infty)$. So, $\phi(x) \geq \phi(1)$ and equality holds if and only if $x = 1$. In other words

$$\frac{x^p}{p} + \frac{1}{q} - x \geq \frac{1}{p} + \frac{1}{q} - 1,$$

$$\frac{x^p}{p} + \frac{1}{q} \geq x.$$

Letting $x = ab/b^q$, we get the Young's inequality. Equality holds if and only if $ab/b^q = 1$, i.e., $a^p = b^q$.

Proposition 1.5.2 (Holders' Inequality). *For* $a, b \in \mathbb{R}^n$, $p > 1$,

$$\sum_{k=1}^{n} |a_k||b_k| \leq \|a\|_p \|b\|_q,$$

where

$$\|a\|_p = \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} \ and \ \|b\|_q = \left(\sum_{k=1}^{n} |b_k|^q\right)^{\frac{1}{q}}.$$

Proof. The inequality clearly holds when $a = (0, \cdots, 0)$. We may assume a 6= $(0, \cdots, 0)$ in the following proof.

By Young's inequality, for each $\varepsilon > 0$ and k,

$$|a_k b_k| = |\varepsilon a_k||\varepsilon^{-1} b_k| \le \frac{\varepsilon^p |a_k|^p}{p} + \frac{\varepsilon^{-q}|b_k|^q}{q}.$$

$$\begin{aligned} \sum_{k=1}^{n} |a_k||b_k| &= |a_1||b_1| + \cdots |a_n||b_n| \\ &\le \frac{\varepsilon^p}{p}\sum_{k=1}^{n}|a_k|^p + \frac{\varepsilon^{-q}}{q}\sum_{k=1}^{n}|b_k|^q \\ &= \frac{\varepsilon^p}{p}\|a\|_p^p + \frac{\varepsilon^{-q}}{q}\|b\|_q^q, \end{aligned} \tag{1.1}$$

for any $\varepsilon > 0$. To have the best choice of ε, we minimize the right hand side of this inequality. Taking derivative of the right hand side of (1.1) as a function of ε, we obtain

$$\varepsilon^{p-1}\|a\|_p^p - \varepsilon^{-q-1}\|b\|_q^q = 0,$$

$$\varepsilon = \frac{\|b\|_q^{\frac{q}{p+q}}}{\|a\|_p^{\frac{p}{p+q}}}.$$

Is the minimum point. (Clearly this function has only one critical point and does not have any maximum.) Plugging this choice of ε into the inequality yields the Holder's inequality after some manipulation.

Proposition 1.5.3 (Murkowski's Inequality). *For a, b* $\in$ F*n and* $p \ge 1$,

$$||a + b||_p \le ||a||_p + ||b||_p.$$

Proof. The inequality clearly holds when p = 1 or ka + bk = 0.

In the following proof we may assume $p > 1$ and $ka + bk > 0$. For each k,

$$|a_k + b_k|^p = |a_k + b_k||\, a_k + b_k\,|^{p-1}$$
$$\le |a_k||\, a_k + b_k\,|^{p-1} + |bk||\, a_k + b_k\,|^{p-1}. \tag{1.2}$$

Applying H¨older's inequality to the two terms on right hand side of (1.2) separately (more precisely, to the pairs of *real* vectors ($|a1|, \cdots,$ $|an|$) and ($|a1 + b1|p-1, \cdots, |an + bn|p-1$), and ($|b1|, \cdots, |bn|$) and

($|a1 + b1|p-1, \cdots, |an + bn|p-1$)), we have and Murkowski's inequality follows.

$$\sum_{k=1}^{n} |a_k + b_k|^p \leq \|a\|_p \left(\sum_{k=1}^{n} |a_k + b_k|^{(p-1)q}\right)^{\frac{1}{q}} + \|b\|_p \left(\sum_{k=1}^{n} |a_k + b_k|^{(p-1)q}\right)^{\frac{1}{q}}$$
$$= (\|a\|_p + \|b\|_p) \left(\sum_{k=1}^{n} |a_k + b_k|^p\right)^{\frac{1}{q}},$$

The last two inequalities allow the following generalization.

1.5.4 Holder's Inequality for Sequences. *For any two sequences a and b in* F, *and p* > 1,

$$\sum_{k=1}^{\infty} |a_k||b_k| \leq \|a\|_p \|b\|_q,$$

where now the summation in the sums on the right runs from 1 to ∞. Since the norms $||a||_p$ and $||b||_q$ are allowed to be zero or infinity, we adopt the convention $0 \times \infty = 0$ in the above inequality.

1.5.5 Murkowski's Inequality for Sequences. For any two sequences a and b in F and p ≥ 1,

$||a + b||_p \leq ||a||_p + ||b||_p,$

where now the summation in the sums runs from 1 to ∞.

1.5.6 Holders' Inequality for Functions. *For p* > 1 *and Riemann integrable functions f and g on* [*a*, *b*], *we have*

$$\int_a^b |fg| \leq \left(\int_a^b |f|^p\right)^{\frac{1}{p}} \left(\int_a^b |g|^q\right)^{\frac{1}{q}}.$$

1.5.7 Murkowski's Inequality for Functions. *For p* ≥ 1 *and Riemann integral functions f and g on* [*a*, *b*], *we have*

$$\left(\int_a^b |f+g|^p\right)^{\frac{1}{p}} \leq \left(\int_a^b |f|^p\right)^{\frac{1}{p}} + \left(\int_a^b |g|^p\right)^{\frac{1}{p}},$$

1.6 NORMED VECTOR SPACES

Let $(X, +, \cdot)$ be a vector space over$\mathbb{F}$. A **norm** on X is a function from X to $[0, \infty)$ satisfying the
following three properties: For all $x, y \in X$ and $\alpha \in F$,

(N1) $||x|| \geq 0$ and "=" holds if and only if $x = 0$,
(N2) $||x + y|| \leq ||x|| + ||y||$,
(N3) $||\alpha x|| = |\alpha|||x||$.

The vector space with a norm, $(X, +, \cdot, ||\cdot||)$, or $(X, ||\cdot||)$, or even stripped to a single X when the context is clear, is called a **normed vector space** or simply a **normed space**. Here are some normed vector spaces.

Example. $(\mathbb{F}^n, || \cdot ||_p)$, $1 \leq p < \infty$, where

$$\|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}$$

Clearly, (N1) and (N3) hold. According to the Murkowski's inequality (N2) holds too. When $p = 2$ and $\mathbb{F}^n = \mathbb{R}^n$ or $\mathbb{C}^n$, the norm is called the **Euclidean norm** or the **unitary norm**.

Example. $(\mathbb{F}^n,, || \cdot ||_\infty)$ where

$$\|x\|_\infty = \max_{k=1,\cdots,n} |x_k|.$$

is called the sup-norm.

Example : Let ℓ^p, $1 \leq p < \infty$, be the collection of all F-valued sequences $x = (x1, x2, \cdots)$ satisfying

$$\sum_{k=1}^{\infty} |x_k|^p < \infty.$$

First of all, from the Murkowski's inequality for sequences the sum of two sequences in ℓ^p belongs to ℓ^p.

With the other easily checked properties, ℓ^p forms a vector space. The function $|| \cdot ||_p$, i.e.

$$\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}}$$

clearly satisfies (N1) and (N3). Moreover, (N2) also holds by Murkowski's inequality for sequences. Hence it defines a norm on ℓ^p.

Example. Let ℓ^∞ be the collection of all F-valued bounded sequences. Define the sup-norm

Clearly ℓ^∞ forms a normed vector space over F

Example. Let $C[a, b]$ be the vector space of all continuous functions on the interval $[a, b]$. For

$1 \leq p < \infty$, define

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}}.$$

By the Murkowski's inequality for functions, one sees that $(C[a, b], k \cdot kp)$ forms a normed space under this norm.

Example: Let $B([a, b])$ be the vector space of all bounded functions on $[a, b]$. The sup-norm

$$\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$$

defines a norm on $B([a, b])$.

Example. In fact, let $F_b(S)$ be the vector subspace of $F(S)$ consisting of all bounded functions from S to F. The sub-norm can be defined on $F_b(S)$ and these examples are special cases obtained by taking different sets S.

Example. Any vector subspace of a normed vector space forms a normed vector space under the same norm. In this way we obtain many

many normed vector spaces. Here are some examples:

The space of all convergent sequences, C, the space of all sequences which converges to 0, C', and the space of all sequences which have finitely many non-zero terms, C'', are normed subspaces of ℓ^∞ under the sup-norm. The space of all continuous functions on $[a, b]$, $C[a, b]$, is an important normed subspace of $B([a, b])$. The spaces $\{f : f(a) = 0, f \in C[a, b]\}$, $\{f : f$ is differentiable, $f \in C[a, b]\}$ and $\{f : f$ is the restriction of a polynomial on $[a, b]\}$ are normed subspaces of $C[a, b]$ under the sup-norm.

But the set $\{f : f(a) = 1, f \in C[a, b]\}$ is not a normed space because it is not a subspace.

To accommodate more applications, one needs to replace $[a, b]$ by more general sets in the examples above. For any closed and bounded subset K in Rn, one may define $C(K)$ to be the collection of all continuous functions in K. As any continuous function in a closed and bounded set must be bounded (with its maximum attained at some point), its sup-norm is well-defined. Thus $(C(K), || \cdot ||_\infty)$ forms a normed space.

On the other hand, let R be any rectangular box in Rn. We know that Riemann integration makes sense for bounded, continuous functions in R. Consequently, we may introduce the normed $k \cdot kp$ = (RR $|f|p$)1/p to make all bounded, continuous functions in R a normed space. However, this p-norm does not form a norm on the space of Riemann integrable functions. Which axiom of the norm is not satisfied?

In addition to above example where new normed spaces are found by restricting to subspaces, there are two more general ways to obtain them. For any two given normed spaces $(X, || \cdot ||_1)$ and $(Y, || \cdot ||_2)$ the function $||(x, y)|| = ||x||_1 + ||y||_2$ defines a norm on the product space $X \times Y$ and thus makes $X \times Y$ the product normed space. On the other hand, to each subspace of a normed space one may form a corresponding quotient space and endow it the quotient norm.

These examples of normed spaces will be used throughout this book. For simplicity the norm of

the space will usually be suppressed. For instance, $\mathbb{F}^n$ always stands for the normed space under the Euclidean or the unitary norm, ℓ^p and ℓ^∞ are always under the p-norms and sup-norm respectively and a single $C(K)$ refers to the space of continuous functions on the closed, bounded set K under the sup-norm.

CHECK YOUR PROGRESS

3.State Murkowski's Inequality for Functions.

4. What is Norm? State its properties

1.7 LET'S SUM UP

In linear algebra, it was pointed out that every vector space has a basis no matter it is of finite or infinite dimension. We studied the concept of Normed spaces, Existence of Basis, Normed vectors. We comprehended three inequalities. Each subspace of a normed space one may form a corresponding quotient space and endow it the quotient norm.

1.8 KEYWORDS

Vector Space- A **vector space** is a set V on which two operations + and · are defined, called **vector** addition and scalar multiplication.

Inequality - In **mathematics**, an **inequality** is a relation which makes a non-equal comparison between two numbers or other **mathematical** expressions

Bounded Function - In mathematics, a **function** f defined on some set X with real or complex values is called **bounded** if the set of its values is **bounded**.

1.9 QUESTION FOR REVIEW

1. Find a relation which satisfies (PO1) and (PO2) but not (PO3), and one which satisfies (PO1) and
(PO3) but not (PO2).

2. Let V be a vector space. Two subspaces U and W form a direct sum of V if for every $v \in V$, there
exist unique $u \in U$ and $w \in W$ such that $v = u + w$. Show that for every subspace U, there exists
a subspace W so that U and W forms a direct sum of V. Suggestion: Try Zorn's lemma.

3. Let $X \times Y$ be the product space of two normed spaces X and Y. Show that it is also a normed
space under the product norm $||(x, y)|| = ||x||_X + ||y||_Y$

4. Give an example to show that $k \cdot kp$ is not a norm on Fn when $n \geq 2$ and $p \in (0, 1)$. Note: In fact, there are reverse Hölder's and Minkowski's inequalities when $p \in (0, 1)$.

1.10 SUGGESTED READINGS

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1.11 ANSWER TO CHECK YOUR PROGRESS

1. Refer explanation -1.2

2. Refer – 1.3

3. Provide representation – 1.4.6

4. Provide explanation -1.4

UNIT 2: NORMED SPACE: ANALYTICAL ASPECTS

STRUCTURE

2.0 OBJECTIVE

Understand the Normed Space as Metric Space

Enumerate the Separability

Comprehend the concept of Completeness

2.1 INTRODUCTION

When a vector space is endowed with a norm, one can talk about the distance between two vectors and consequently it makes sense to talk about limit, convergence and continuity. The underlying structure is that of a metric space. We give a brief introduction to metric space in the first section and use it to discuss three analytical properties of a normed vector space, namely, separability, completeness and Bolzano-Weierstrass property, in later sections. Emphasis is on how these properties are preserved, modified or lost when one passes from finite to infinite dimensions. Our discussion on metric spaces is minimal in order to avoid possible overlap with a course on point set topology.

2.2 NORMED SPACE AS METRIC SPACE

Let M be a non-empty set. A function $d : M \times M 7\rightarrow [0, \infty)$ is called a **metric** on M if $\forall p, q, r \in M$

(D1) $d(p, q) \geq 0$, and "=" holds if and only if $p = q$.
(D2) $d(p, q) = d(q, p)$.
(D3) $d(p, q) \leq d(p, r) + d(r, q)$.

The pair (M, d) is called a **metric space**.

To be precise, we have

• Let $\{pn\}$ be a sequence in (M, d). We call $p \in M$ the **limit** of $\{pn\}$ if for any $\varepsilon > 0$, there exists n_0 such that $d(pn, p) < \varepsilon$ for all $n \geq n_0$. Write $p = \lim n\rightarrow\infty\ pn$ or simply $pn \rightarrow p$..

• The sequence $\{pn\}$ is called a **Cauchy sequence** if for any $\varepsilon > 0$, there exists n_0 such that
$d(p_n, p_m) < \varepsilon$, for all $n, m \geq n_0$.

• Let $f : (M, d) \mapsto (N, \rho)$ where (N, ρ) is another metric space be a function and $p_0 \in M$. f is continuous at p_0 if $f(p0) = \lim_{n\rightarrow\infty} f(p_n)$ whenever $\lim_{n\rightarrow\infty} p_n = p$. Alternatively, for any $\varepsilon > 0$, it is required that there exists $\delta > 0$ such that $\rho(f(p), f(p_0)) < \varepsilon$ whenever $d(p, p_n) < \delta$. f is called a continuous function on M if it is continuous at every point. Very often it is more convenient to use the language of topology (open and closed sets) to describe these concepts. To introduce it lets denote the metric ball centered at p, $\{q \in M : d(q, p) < r\}$, by $B_r(p)$. A non-empty subset G of M is called an **open set** if $\forall p \in G$, there exists a positive r (depending on p) such that $Br(p) \subset G$. We define the empty set to be an open set.

Also the whole M is open because it contains every ball. It is easy to see that any metric ball $B_R(p_0)$, is an open set. For, let $p \in B_R(p_0)$, we claim that $Br(p)$, $r = R - d(p, p_0)$, is contained inside $B_R(p_0)$. This is a consequence of the triangle inequality (D3): Let $q \in B_r(p)$, then $d(q, p_0) \leq d(q, p) + d(p, p_0) < r + d(p, p_0) = R$, so $q \in B_R(p_0)$. Roughly speaking, an open set is a set without boundary. A subset E is called a **closed set** if its complement $M \setminus E$ is an open set. The empty set is a closed set as its complement is the whole space. By the same reason M is also closed. So the empty set and the whole space are both open and closed.

Proposition 2.1.1 Let (M, d) be a metric space. Then the union of open sets and the intersection of finitely many open sets are open. The intersection of closed sets and the union of finitely many closed sets are closed.

Proof. That any countable or uncountable open sets still form an open set comes from definition. As for finite intersections, let $G = \cap_{k=1}^{n} Gk$ where Gk is open. For $x \in G \subset Gk$, we can find a metric ball $B_{rk}(x) \subset Gk$ for each k since Gk is open. It follows that the ball $Br(x)$, $r = \min\{r1, r2, \cdots, r_n\}$ is contained in G, so G is open.

The assertions on closed sets come from taking complements of the assertions on open sets. Notice that infinite intersection of open sets may not be open. Let us consider the open intervals $I_n = (-1/n, 1 + 1/n)$, $n \in \mathbb{N}$ in $\mathbb{R}$ under the Euclidean metric. Then $\cap I_n = [0, 1]$ which is not open. Similarly, let $\{a1, a2, a3, \cdots, \}$ be an enumeration of all rational numbers and set $Fn = \{a1, a2, \cdots, an\}$. Then each Fn is closed, but $\cup Fn$ is the set of all rational numbers which is clearly not closed in $\mathbb{R}$.

To have a better picture about the closed set we introduce the notion of a the limit point of a set.

We call a point $p \in M$ a **limit point** of a set E if for all $r > 0$, $Br(x) \setminus \{x\} \cap E \neq \emptyset$. The limit point is related to a set, but the limit, although it is also a point, is related to a sequence. They are not the same.

For example, consider the sequence $\{1, 1\ 2, 1\ 3, 1\ 4, \cdots\}$, its limit is clearly 0. If we regard $\{1, 1\ 2, 1\ 3, 1\ 4, \cdots\}$ as a set, 0 is its unique limit point. However, for the sequence $\{0, 2, 1, 1, 1, \cdots\}$, the limit is 1 but as a set it has no limit point.

Proposition 2.2.2. *A non-empty set E is closed if and only if it contains all its limit points.*

Proof. Let E be a closed set. By definition $M \setminus E$ is open. If $p \in M \setminus E$, there exists r such that

$B_r(p) \subset M \setminus E$, ie, $Br(p)$ T $E = \emptyset$. It follows that p cannot be a limit point of E. This shows that any limit point of E must belong to E.

Conversely, we need to show $M \setminus E$ is open. Since E already contains all limit points, any point

$p \in M \setminus E$ cannot be a limit point of E. Therefore, there is an r such that $Br(p)$ T $E = \emptyset$, but that

means $Br(p) \subset M \setminus E$, so $M \setminus E$ is open.

The **closure** of E, denoted by $\bar{E}$, is defined to be the union of E and its limit points. By Proposition 2.1.1 it is easily shown that $\bar{E}$, is the smallest closed set containing E, that is, $\bar{E} \subset F$ whenever F is a closed set containing E.

In terms of the language of open-closed sets (or topology), a sequence $\{x_n\} \to x$ can be expressed as, for each open set G containing x, there exists an n_0 such that $x_n \in G$ for all $n \geq n_0$.

For $f : (M, d) \mapsto (N, \rho)$ where (N, ρ) is another metric space. In terms of topology, we have the following characterization of continuity:

Proposition 2.2.3. f : $(M, d) \mapsto (N, \rho)$ *is continuous if and only if* $f^{-1}(G)$ *is open for any open* G *in* N.

Proof. Assume on the contrary that there is an open set G in N whose pre-image is not open. We can find some $p_0 \in f^{-1}(G)$ and $pn \in M \setminus f$ $-1(G)$ with $\{pn\} \to p_0$. By continuity, $\{f(pn)\} \to f(p0)$. As G is open, there exists some $n0$ such that $f(pn) \in G$ for all $n \geq n0$. But this means that $f^{-1}(G)$ contains pn for all $n \geq n_0$, contradiction holds. We conclude that $f -1(G)$ must be open when G is open.

On the other hand, suppose f is not continuous, then there exists $\{pn\} \to$ $p0$ in M but $\{f(pn)\}$ does

not converge to $f(p0)$. Then there exists $\rho > 0$ and a subsequence $\{f(pnj$ $)\}$, $f(pnj\ ) \notin B\rho(f(p0))$, $\forall nj$. As $B\rho(f(p_0))$ is open, $f -1(B\rho(f(p0)))$ is open in M, so there exists $n0$ such that $pn \in f -1(B\rho(f(p_0)))$.

But then $f(pn) \in B\rho(f(po))$ for all $n \geq n0$, contradiction holds. Let E be any nonempty subset of (M, d). Then it is clear that (E, d) forms a metric space. It is called a metric subspace or simply a subspace. As we will see, the subspaces formed by closed subsets are particularly important since they inherit many properties of M.

Now, let us return to normed spaces. Let (X, $k \cdot k$) be a normed space.

Define $d(x, y) = ||x - y||$.

Using (N1)-(N3), it is easy to verify (D1)-(D3) hold for d, so (X, d) becomes a metric space. This metric is called the induced metric of the norm $|| \cdot ||$. Of course, there are many metrics which are not induced by norms. But in functional analysis most metrics are induced in this way.

¿From now on whenever we have a normed space, we can talk about convergence and continuity implicitly referring to this metric.

The following statements show that the norm and the algebraic operations on the vector spaces interact nicely with the metric.

Proposition 2.2.4. Let (X, $\| \cdot \|$) be a normed space. Then

(a) The norm $\| \cdot \|$ is a continuous function from X to $[0, \infty)$;

(b) Addition, as considered as a map X × X 7→ X, and scalar multiplication, a map F × X 7→ X, are continuous in X × X and F × X respectively.

Proof. (a) $p_n \to p$ means $d(p_n, p) \to 0$. But then

$||p_n - p|| = d(p_n, p) \to 0$, $\big| ||p_n|| - ||p|| \big| \leq ||p_n - p|| \to 0$.

(b) We need to show $p_n \to p$ and $q_n \to q$ implies $p_n + q_n \to p + q$. This is clear from $d(p_n + q_n, p + q) = ||(p_n + q_n) - (p + q)|| \leq ||p_n - p|| + ||qn - q||$ $= d(p_n, p) + d(qn, q)$.

For scalar multiplication, need to show $\alpha_n \to \alpha$ and $p_n \to p$ implies $\alpha_n p_n \to \alpha_p$. By (a), we have

$||p_n|| \to ||p||$. Hence for $\varepsilon = 1$, there exists some $n0$ such that $|||pn|| - ||p||| < 1$, or $||p_n|| \leq 1 + ||p||$, for all $n \geq n_0$.

$$\begin{aligned} d(\alpha_n p_n, \alpha p) &= \|\alpha_n p_n - \alpha p\| = \|\alpha_n p_n - \alpha p_n + \alpha(p_n - p)\| \\ &\leq |\alpha_n - \alpha|\|p_n\| + |\alpha|\|p_n - p\| \\ &\leq |\alpha_n - \alpha|(\|p\| + 1) + |\alpha|\|p_n - p\| \\ &= |\alpha_n - \alpha|(\|p\| + 1) + |\alpha| d(p_n, p) \to 0 \end{aligned}$$

as $n \to \infty$.

As an interesting application of the continuity of norm, we study the equivalence problem for norms. Consider two norms defined on the same space $(X, ||\cdot||_1)$ and $(X, ||\cdot||_2)$. We call $||\cdot||_2$ is **stronger** than $||\cdot||_1$ if there exists $C > 0$ such that

$$||\cdot||_1 \leq C||x||_2, \forall x \in X.$$

In particular, it means $xn \to x$ in $||\cdot||_2$ implies $xn \to x$ in $k \cdot k1$. Two norms are **equivalent** if $||\cdot||_1$ is stronger than $k \cdot k2$ and $k \cdot k2$ is stronger than $k \cdot k1$. In other words, there exists $C_1, C_2 > 0$ such that

$$C_1||x||_2 \leq ||x||_1 \leq C_2||x||_2, \forall x \in X.$$

Example: On $\mathbb{F}^n$ consider the p-metric $d_p(x, y) = ||x - y||_p$ induced from the p-norm $(1 \leq p \leq \infty)$.

Theorem 2.2.5. *Any two norms on a finite dimensional space are equivalent.*

Proof. In the following proof we assume the space is over $\mathbb{R}$. The same arguments work for spaces over $\mathbb{C}$.

Step 1: Take $X = \mathbb{R}^n$ first. It suffices to show that any norm on $\mathbb{R}^n$ is equivalent to the Euclidean norm.

Let $||\cdot||$ be a norm on $\mathbb{R}^n$. For $x = \sum \alpha_j e_j$, recalling that $||x||_2 = \sqrt{\sum |\alpha_j|^2}$, we have

$$\|x\| \leq \sum |\alpha_j|\|e_j\| \leq \sqrt{\sum |\alpha_j|^2}\sqrt{\sum \|e_j\|^2} = C\|x\|_2,$$

where $C = (\sum_j ||e_j||^2)^{1/2}$. This shows that $||\cdot||_2$ is stronger than $||\cdot||$. To establish the other inequality, letting $\phi(x) = ||x||$, from the triangle inequality $|\phi(x) - \phi(y)| \leq ||x - y|| \leq C||x - y||_2$ φ is a continuous function with respect to the Euclidean norm.

Consider

$$\alpha \equiv \inf\{\varphi(x) : x \in \mathbb{R}^n, \|x\|_2 = 1\}.$$

As the function ϕ is positive on the unit sphere of $||\cdot||_2$, α is a non-negative number. The second inequality will come out easily if α is positive. To see this we observe that for every nonzero $x \in \mathbb{R}^n$

$$0 < \alpha \leq \varphi(\frac{x}{\|x\|_2}) = \frac{\|x\|}{\|x\|_2},$$

$$\alpha\|x\|_2 \leq \|x\|, \ \forall x.$$

To show that α is positive, we use the fact that every continuous function on a closed and bounded subset of $\mathbb{R}^n$ must attain its minimum. Applying it to φ and the unit sphere $\{||x||_2 = 1\}$, the infimum α is attained at some point x_0 and so in particular $\alpha = \varphi(x_0) > 0$.

Step 2: For any n dimensional space X, fix a basis $\{x1, x2, \cdots, xn\}$. For any $x \in X$, we have a unique representation $x = \sum_{k=1}^{n} a_k x_k$. The map x 7→ $(\alpha1, \cdots, \alpha n)$ is a linear isomorphism from X to $\mathbb{R}^n$. Any norm $|| \cdot ||$on X induces a norm $|| \cdot \tilde{||}$ on $\mathbb{R}^n$ by

$$\|(\alpha_1, \cdots, \alpha_n)\tilde{\|} = \|\sum \alpha_k x_k\|.$$

Let $|| \cdot ||_1$ and $|| \cdot ||_2$ be two norms on X and let $|| \cdot \tilde{||}_1$and $|| \cdot \tilde{||}_2$be the corresponding norms on $\mathbb{R}^n$. From Step 1, there exist $C1, C2 > 0$ such that

$$C_1\|x\|_2 = C_1\|\alpha\tilde{\|}_2 \leq \|\alpha\tilde{\|}_1 = \|x\|_1 \leq C_2\|\alpha\tilde{\|}_2 = C_2\|x\|_2.$$

Example: Consider the norms $|| \cdot ||_\infty$ and $|| \cdot ||_1$ on $C[a, b]$. On one hand, from the obvious estimate

$$\|f - g\|_1 = \int_a^b |f - g|(x)dx \leq (b - a)\|f - g\|_\infty$$

we see that $|| \cdot ||_\infty$ is stronger than $|| \cdot ||_1$. But they are not equivalent. It is easy to find a sequence of functions $\{fn\}$ in $C[a, b]$ which satisfies $||f_n||_\infty = 1$ but $||f_n||_1 \to 0$. Consequently, it is impossible to find a constant C such that $||f_k||_\infty \leq C||f_k||_1$ for all f. In other words, $||\cdot ||_1$ cannot be stronger than $|| \cdot ||_\infty$

Notes

2.3 SEPARABILITY

There are some important and basic properties of the space of all real numbers which we would like to study in a general normed space. They are

- Separability
- Completeness
- Bolzano-Weierstrass property.

As we all know, the rational numbers are dense in the space of all real numbers. The notion of a dense set makes perfect sense in a metric space. A subset E of (M, d) is a **dense set** if its closure is the whole M, or equivalently, for every $p \in M$, there exists $\{pn\}$ in E, $pn \to p$. A metric space is called **separable** if it has a countable dense subset.

Thus R is separable because it contains the countable dense subset Q.

The following two results show that there are many separable normed spaces.

Proposition 2.3.1. *The following normed spaces are separable:*

(a) (Fn, $k \cdot kp$) $(1 \leq p \leq \infty)$,

(b) \`p $(1 \leq p < \infty)$,

(c) $(C[a, b], k \cdot kp)$ $(1 \leq p \leq \infty)$.

Proof. We only prove (c) and leave (a) and (b) to you. For any continuous, real-valued f, given any $\varepsilon > 0$, by Weierstrass approximation theorem there exists a polynomial p such that $||f - p||_\infty < \varepsilon$.

The coefficients of p are real numbers in general, but we can approximate them by rational numbers, so without loss of generality we may assume its coefficients are rational. The set $E = \{p \in C[a, b] : p$ is a polynomial with rational coefficients$\}$ forms a countable, dense subset of $(C[a, b], || \cdot ||_\infty)$.

For any finite p, we observe that

$$\|f\|_p = (\int_a^b |f(x)|^p \, dx)^{\frac{1}{p}} \leq (b-a)^{\frac{1}{p}} \|f\|_\infty.$$

As for every f, there exists $p_n \in E$, $||p_n - f||_\infty \to 0$, we also have $||p_n - f||_p \to 0$, so E is also dense in $(C[a, b], || \cdot ||_p)$.
When the function is complex-valued, simply apply the above result to its real and imaginary parts.
Proposition 2.3.2. Any subset of a separable metric space is again separable.

Proof. Let $Y \subset X$ and E a countable, dense subset of X. Write $E = \{p_k\}_{k=1}^{\infty}$.For each m, $B_{1/m}(p_k) \cap Y$ may or may not be empty. Pick a point $p_{m,k}$ if it is not empty. The collection of all these $p_{m,k}$ points forms a countable subset S of Y. We claim that it is dense in Y. For, any $p \in Y$, and $m > 0$, there exists $p_k \in B_{1/m}(p)$, $p_k \in E$ by assumption. But then $p \in B_{1/m}(p_k)$ which means $B_{1/m}(pk) \cap Y \neq \phi$. Then we have $pm,k \in B\ 1/m(p_k)$ and so $d(p, p_{m,k}) \leq d(p, pk) + d(pk, p_{m,k}) < 2/m$.
Now we give an example of a non-separable space.

Proposition 2.3.3. ℓ^∞ is not separable.

Proof. Consider the subset F of ℓ^∞ consisting of all sequences of the form $(a1, a2, a3, \cdots)$ where $a_k = 1$ or 0. In view of Proposition 2.2.2, it suffices to show that F is not separable. First of all, it is an uncountable set as easily seen from the correspondence
$(a1, a2, a3, \cdots) \leftrightarrow 0.a1a2a3 \cdots$ in binary representation of a real number

which maps F onto $[0, 1]$.
For each $x, y \in F$, we have

$d(x, y) = ||x - y||\infty = \sup k\ |x_k - y_k| = 1$.
It follows that the balls $B1/2(x)$, $x \in F$, are mutually disjoint. Let E be a dense set in F. By definition there exists some $p_x \in B_{1/2}(x) \cap E$. Since

these balls are disjoint, all px are distinct, so $\{px\}$ forms an uncountable subset of E. Thus E is also uncountable. We have shown that there are no countable dense subsets in F , that is, F is not separable.

CHECK YOUR PROGRESS

1.Define Dense set & separable

2. Prove - ℓ^∞ *is not separable*

2.4 COMPLETENESS

A metric space (M, d) is **complete** if every Cauchy sequence converges. As we all know, R is a complete metric space.

Proposition 2.4.1. *The following spaces are complete:*

(a) $(\mathbb{F}^n, \|\cdot\|_p)$ $(1 \le p \le \infty)$,

(b) ℓ^p $(1 \le p \le \infty)$,

(c) $(C[a, b], \|\cdot\|_\infty)$.

Proof. (a) Let $\{p_k\}$ be a Cauchy sequence in $\mathbb{F}^n$. For $p^k = (p_1^k, \cdots, p_n^k)$, from

$$|p_j^k - p_j^l| \le \|p^k - p^l\|_p$$

we see that $\{p_j^k\}$ is a Cauchy sequence in F for each $j = 1, 2, \cdots, n$. By the completeness of F there exists p_j such that $p_j^k \to pj$ as $k \to \infty$ for each j. Given $\varepsilon > 0$, there exists k_0 such that

$$|p_j^k - p_j| \le \varepsilon, \quad \forall k \ge k0.$$

Summing up over j, $||p^k - p||_p < n^{1/p} \max_j |p_{kj} - p_j| < n^{1/p}\varepsilon$, $\forall k \geq k_0$, which shows that $p^k \to p \equiv (p_1, \cdots, p_n)$. We leave the proofs of (b) and (c) to the reader. Note that (c) was a theorem on uniform convergence in elementary analysis.

But $C[a, b]$ is not complete in the L^p-norm ($1 \leq p < \infty$). To find a divergent Cauchy sequence we let

$$\varphi_n(x) = \begin{cases} 1, & x \in [-1, 0] \\ -nx + 1, & x \in [0, \frac{1}{n}] \\ 0, & x \in [\frac{1}{n}, 1]. \end{cases}$$

$$\varphi(x) = \begin{cases} 1, & x \in [-1, 0] \\ 0, & x \in (0, 1]. \end{cases}$$

It is easy to see that $||\phi_n - \phi||_p \to 0$.

Therefore,

$$\|f_n - f_m\|_p \leq \|f_n - f\|_p + \|f_m - f\|_p \to 0,$$

as $n, m \to \infty$, that is, $\{f_n\}$ is a Cauchy sequence in p-norm. To show that it does not converge to a continuous function let's assume on the contrary it converges to some continuous f. From as $n \to \infty$, we see that f is identical to ϕ on $[-1, 0]$ since both functions are continuous on $[-1, 0]$. In particular, $f(0) = 1$, so by continuity $f > 0$ on $[0, \delta]$ for some positive δ. However, since f and ϕ are continuous on $[\delta, 1]$ by a similar argument as above f is identical to ϕ on $[\delta, 1]$, but then $g(\delta) = \phi(\delta) = 0$,contradiction holds.

Fortunately, one can make any metric space complete by putting in ideal points. In general, a map $f : (M, d) \longrightarrow (N, \rho)$ is called a **metric preserving map** if $\rho(f(x), f(y)) = d(x, y)$ for all x, y in M.

Note that a metric preserving map is necessarily injective. In some texts the name an **isometric** is used instead of a metric preserving map. We prefer to use the former and reserve the latter for a metric preserving and subjective map.

Notes

A complete metric space $(\widetilde{M}, \tilde{d})$ is called the **completion** of a metric space (M, d) if there exists an metric preserving map Φ of M into $\widetilde{M}$ such that $\Phi(M)$ is dense in $\widetilde{M}$.

Theorem 2.4.2. *Every metric space has a completion.*

Proof. Let $\mathcal{C}$ be the collection of all Cauchy sequences in (M, d). We introduce a relation $\sim$ on C by $x \sim y$ if and only if $d(xn, yn) \to 0$ as $n \to \infty$. It is routine to verify that $\sim$ is an equivalence relation on C . Let M f = $C / \sim$ and define a map: $\widetilde{M} \times \widetilde{M} \mapsto [0, \infty)$ by

$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} d(x_n, y_n)$$

where $x = (x1, x2, x3, \cdots)$ and $y = (y1, y2, y3, \cdots)$ are respective representatives of x e and y e. We note that the limit in the definition always exists: For

$$d(x_n, yn) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n)$$

and, after switching m and n,

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_m, y_n).$$

As x and y are Cauchy sequences, $d(xn, xm)$ and $d(ym, yn) \to 0$ as $n, m \to \infty$, so $\{d(xn, yn)\}$ is a Cauchy sequence of real numbers.

Step 1. Well-definedness of $\tilde{d}$. To show that $\tilde{d}\,(\tilde{x}, \tilde{y})$ is independent of their representatives let $x \sim x'$ and $y \sim y'$. We have

$$d(x_n, y_n) \leq d(x_n, x'_n) + d(x'_n, y'_n) + d(y'_n, y_n).$$

After switching x and $x0$, and y and $y0$,

$$|d(x_n, y_n) - d(x'_n, y'_n)| \leq d(x_n, x'_n) + d(y_n, y'_n).$$

As $x \sim x'$and $y \sim y'$, the right hand side of this inequality tends to 0 as $n \to \infty$. Hence $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x_n', y_n')$.

Step 2. $\tilde{d}$ is a metric. This is straightforward and is left as an exercise.

Step 3. Φ is metric preserving and has a dense image in $\widetilde{M}$. More precisely, we need to show that there is a map $\Phi : M \mapsto \widetilde{M}$ so that $\tilde{d}\,(\Phi(x), \Phi(y)) = d(x, y)$ and $\Phi(M)$ is dense in $\widetilde{M}$.

Given any x in M, the "constant sequence" $(x, x, x, \cdots)$ is clearly a Cauchy sequence. Let $\tilde{x}$ be its equivalence class in $\mathcal{C}$. Then $\Phi x = \tilde{x}$ e defines a map from M to $\widetilde{M}$. Clearly

$$\tilde{d}(\Phi(x), \Phi(y)) = \lim_{n\to\infty} d(x_n, y_n) = d(x, y)$$

since $x_n = x$ and $y_n = y$ for all n, so Φ is metric preserving and it is injective in particular. To show that $\Phi(M)$ is dense in $\widetilde{M}$ we observe that any x e in M f is represented by a Cauchy sequence $x = (x1, x2, x3, \cdots)$. Consider the constant sequence $x^n = (x_n, x_n, x_n, \cdots) \in \Phi(M)$. We have

$$\tilde{d}(\tilde{x}, \tilde{x}_n) = \lim_{m\to\infty} d(x_m, x_n).$$

Given $\varepsilon > 0$, there exists n' such that $d(x_m, x_n) < \varepsilon/2$ for all $m, n \geq n'$. Hence $\tilde{d}\,(\tilde{x}, \tilde{x}_n) = \lim_{m\to\infty} d(x_m, x_n) < \varepsilon$ for $n \geq n'$. That is $\tilde{x}^n \to \tilde{x}$ as $n \to \infty$, so $\Phi(M)$ is dense in M.

Step 4. $\tilde{d}$ is a complete metric on $\widetilde{M}$. Let $\{x$ e$n\}$ be a Cauchy sequence in $\widetilde{M}$. As $\Phi(M)$ is dense in $\widetilde{M}$, for each n we can find a $\widetilde{y^n}$ in $\Phi(M)$ such that

$$\tilde{d}(\tilde{x}^n, \tilde{y}^n) < \frac{1}{n}.$$

So $\{\widetilde{y^n}\}$ is Cauchy in $\tilde{d}$. Let y_n be the point in M so that $y^n = (y_n, y_n, y_n, \cdots)$ represents $\widetilde{y^n}$. Since Φ is metric preserving and $\{\widetilde{y^n}\}$ is Cauchy in $\tilde{d}$, $\{yn\}$ is a Cauchy sequence in M. Let $(y1, y2, y3, \cdots) \in y$ e in $\widetilde{M}$.. We claim that $\tilde{y}. = \lim_{n\to\infty} \widetilde{x^n}$. in $\widetilde{M}$. For, we have

$$\begin{aligned} \tilde{d}(\tilde{x}^n, \tilde{y}) &\leq \tilde{d}(\tilde{x}^n, \tilde{y}^n) + \tilde{d}(\tilde{y}^n, \tilde{y}) \\ &\leq \frac{1}{n} + \lim_{m\to\infty} d(y_n, y_m) \to 0 \end{aligned}$$

as $n \to \infty$.

The idea of this proof is due to Cantor, who used equivalence classes of Cauchy sequences of rational numbers to construct real numbers.

Another popular approach for the real number system is by “Dedekind cut”.

The uniqueness of the completion could be formulated as follows. Let Ψ_i : $(M, d) \to (M_i, d_i)$, $i = 1, 2$, be two metric preserving maps with dense images. Then the map Ψ2Ψ− 1 1 : Ψ1(*M*1) → *M*2 can be extended to be an isometry between M_1 and M_2.

CHECK YOUR PROGRESS

3. Explain Completness

4. Explain - *Every metric space has a completion.*

2.5 LET’S SUM UP

When a given metric space is induced from a normed space, it is naturally to ask it is possible to make the completion into a normed space so that the complete metric is induced by the norm on the completion.

2.6 KEYWORDS

Relation: A **relation** is a **relationship** between sets of values. In **math**, the **relation** is between the x-values and y-values of ordered pairs. The set of all x-values is called the domain, and the set of all y-values is called the range.

Convergence, in mathematics, property (exhibited by certain infinite series and functions) of approaching a limit more and more closely as an argument (variable) of the function increases or decreases or as the number of terms of the series increases

Completeness- every nonempty set that has an upper bound has a smallest such bound, a property not possessed by the rational numbers.

2.7 QUESTION FOR REVIEW

1. Let $p \in M$. Show that f is continuous at p if and only if for every open set G in N containing $f(p)$, there exists an open set U in M containing p such that $U \subset f^{-1}(G)$.

2. Show that f is continuous in M if and only if for every closed set F in N, $f^{-1}(F)$ is a closed set in M.

3. Show that in a normed space the closed metric ball with radius R centered at x, $\{y: d(y, x) \leq R\}$,
is the closure of the open metric ball $BR(x)$.

4. Show that any finite dimensional subspace of a normed space is closed. Can you find a subspace which is not closed, say, in ℓ^1? How about in $C[0, 1]$?

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2.9 ANSWER TO CHECK YOUR PROGRESS

1. Refer explanation -2.2

2. Provide proof – 2.2.3

3. Provide explanation– 2.3 & statement of theorem & proof -- 2.3.1

4. Provide proof -2.3.2

UNIT 3: NORMED SPACE: ANALYTICAL ASPECTS II

STRUCTURE

3.0 OBJECTIVE

Understand the concept of Uniform Boundedness Principle

Comprehend Open Mapping Theorem

Enumerate the concept of Spectrum

3.1 INTRODUCTION

In mathematics, a **normed vector space** is a vector space over the real or complex numbers, on which a **norm** is defined. A norm is the formalization and the generalization to real vector spaces of the intuitive notion of "length" in the real world.

3.2 COMPLETENESS THEOREM

Theorem 3.1.1. *Let* $(X, k \cdot k)$ *be a normed space and* $\tilde{X}$ *its completion under the induced metric of* X*. There is a unique normed space structure* *on $\tilde{X}$ so that the quotient map $x \mapsto \tilde{x}$ becomes linear and norm-preserving. Moreover, the metric induced by this norm $\tilde{X}$ is identical to the completion metric.*

Proof. We only give the outline of the proof

Step 1. Let $\Phi : X \to \tilde{X}$ be the quotient map. As $\Phi(X)$ is dense in $\tilde{X}$, for any $\tilde{x}$, $\tilde{y}$ in $\tilde{X}$, we can find sequences $\{\widetilde{x^n}\}$, $\{\widetilde{y^n}\}$ converging to $\tilde{x}$, $\tilde{y}$ respectively. We define an addition and a scalar multiplication on $\tilde{X}$ by

$$\tilde{x} + \tilde{y} \equiv \lim_{n\to\infty} \Phi(x_n + y_n),$$

$$\alpha\tilde{x} \equiv \lim_{n\to\infty} \Phi(\alpha x_n),$$

where x_n and y_n are representatives of x f n and y f n respectively. You need to establish three things.

First, these operations are well-defined, that's, they are independent of the representatives.

Second, they make $\tilde{X}$ into a vector space.

Third, the map Φ is linear from X to $\tilde{X}$. (In fact, this follows immediately from the definitions.)

Step 2. Introduce a map on $\tilde{X}$ by

$$||\tilde{x}\,|| \equiv \tilde{d}\,(\tilde{x},0).$$

Then verify the following three facts:

First, translational invariance: $\tilde{d}\,(\tilde{x} + \tilde{y}, \tilde{y}) = \tilde{d}\,(\tilde{x}, 0)$ for all $\tilde{x}$and $\tilde{y}$.

Second, use translational invariance to show that this map really defines a norm on $\tilde{X}$.

Third, show that the metric induced by this norm coincides with the completion metric. This in fact follows from the definition of the norm.

Step 3. Show that if there is another normed space structure on $\tilde{X}$ so that the quotient map Φ becomes linear and norm-preserving, then this normed space structure is identical to the one given by Steps 1 and 2. Essentially this follows from the fact that $\Phi(X)$ is dense in $\tilde{X}$.

As an immediate application of these results, we let $L^p(a, b)$ be the completion of $C[a, b]$ under the L^p-norm. We shall call an element in $Lp(a, b)$ an Lp-function, although it makes sense only when the element is really in $C[a, b]$. Such terminology is based on another construction in real analysis where we really identify $L^p(a, b)$ as the function space consisting of Lp-integrable functions. We do not need this fact in this course.

A complete normed space is called a **Banach space**. Banach space is one of the fundamental concepts in functional analysis. Now we know that even a space is not complete, we can make it into a Banach space. The following nice properties of Banach spaces hold:

• Any closed subspace of a Banach space is a Banach space.

• The product space of two Banach spaces is a Banach space under the product norm.

• For any closed subspace Z of a Banach space X, the quotient space X/Z is a Banach space under
the quotient norm.

We have shown that the spaces $\mathbb{F}n$, $\ell^p(1 \leq p \leq \infty)$, $C[a, b]$ and $L^p[a, b]$, $p \in [1, \infty)$, are Banach spaces. In fact, for any metric space X, the space $C_b(X) = \{f : f$ is bounded and continuous in $X\}$
forms a Banach space under the sup-norm. For any measure space (X, μ), the space $Lp(X, \mu) = \{f : f$ is L^p-integrable$\}$ forms a Banach space under the L^p-norm. Finally, without requiring any topology or integrability on the functions, the space $L\infty(X)$ consisting of all bounded functions in a nonempty set X is a Banach space under the sup-norm.

So far we have encountered three types of mathematical structure, namely, those of a vector space, a metric space and a normed space. How do we identify two spaces from the same structure? Well, first of all, we view two vector spaces the same if there exists a bijective linear map between them. A bijective linear map is also called a linear isomorphism. Next, two metric spaces are the same if there exists a metric preserving bijective map, that is, an isometry, from one to the other. Finally, two normed spaces are the same if there exists a norm-preserving linear isomorphism from one to the other.

3.3 SEQUENTIAL COMPACTNESS

In the space of real numbers, any bounded sequence has a convergent subsequence. This property is called the Bolzano-Weierstrass property. In a general setting, it is more convenient to put this concept in another way. Let *E* be a subset of the metric space (*M*, *d*). *E* is called **sequentially compact** if every sequence in *E* enjoys the Bolzano-Weierstrass property, that is, it contains a convergent subsequence, in *E*. Any sequentially compact set is necessarily a closed set. It is clear that the Bolzano-Weierstrass property essentially refers to the fact that the interval [*a*, *b*] is sequentially compact in ℝ. The same as in the case of R, one can show that every closed and bounded set in $\mathbb{R}^n$ is sequentially compact. Surprisingly, this property is a characterization of finite dimensionality.

Theorem 3.3.1. *Any closed ball in a normed space is sequentially compact if and only if the space is of finite dimension.*

Lemma 3.2.2. *Let Y be any proper finite dimensional subspace of the normed space* (*X*, *k* · *k*). *Then for any x* ∈ *X* \ *Y*, *there exists y*0 ∈ *Y such that*

$$d \equiv dist(x, Y) \equiv \inf_{y \in Y} \|x - y\| > 0.$$

is realized at y'.

The distance *d* is positive because *Y* is closed due to finite dimensionality and *x* stays outside *Y* .

Proof. Let $\{y_k\}$ be a minimizing sequence of the distance, that is, $d = \lim_{k\to\infty} ||x - y_k||$. We may assume $||x - y_k|| \le d + 1$, for all *k*.

Then

$$||y_k|| \le ||x|| + ||y_k - x|| \le ||x|| + d + 1$$

which means that *{yk}* is a bounded sequence in *Y* . Since *Y* is finite dimensional, it is closed and Bolzano-Weierstrass property holds in it, there exists a subsequence $\{y_{nj}\}$ converging to some y_0 in Y.

We have $d = \lim_{nj\to\infty} ||x - y_{nj}|| = ||x - y0||$, hence y' realizes the distance between x and Y.

Proof of Theorem 3.2.1 It suffices to show that the closed unit ball $\{x \in X : ||x|| \leq 1\}$ is not sequentially compact when X is of infinite dimension. Let $\{x1, x2, x3, \cdots\}$ be a sequence of linearly independent vectors in X. We are going to construct a sequence $\{z_n\}$, $z_n \in \langle x_1, x_2, \cdots, x_n\rangle$, $||z_n|| = 1$ satisfying that $||z_n - x|| \geq 1$, for all $x \in \langle x_1, x_2, \cdots, x_{n-1}\rangle$, $n \geq 2$.

Set $z1 = x1/kx1k$. For $xn \in h / x1, x2, \cdots, xn-1i$, $n \geq 2$, let $yn-1$ be the point in $hx1, x2, \cdots, xn-1i$ realizing $dist(xn, \langle x_1, \cdots, x_{n-1}\rangle)$. Let

$$z_n = \frac{x_n - y_{n-1}}{||x_n - y_{n-1}||}.$$

Then $||z_n|| = 1$ and, for all $y \in \langle x_1, \cdots, x_{n-1}\rangle$,

$$||z_n - y|| = \left\|\frac{x_n - y_{n-1}}{||x_n - y_{n-1}||} - y\right\| = \frac{||x_n - y'||}{||x_n - y_{n-1}||} \geq 1,$$

where $y' = y_{n-1} + ||x_n - y_{n-1}||\, y \in \langle x_1, \cdots, x_{n-1}\rangle$, since $||x_n - y_{n-1}|| \leq ||x_n - y'||$.

We claim that the bounded sequence $\{z_n\}$ does not have a convergent subsequence. For, if it has, this subsequence is a Cauchy sequence. Taking $\varepsilon = 1$, we have

$$||z_{nk} - z_{nj}|| < 1, \quad k, j \text{ sufficiently large.}$$

Taking $n_k > n_j$, as $||z_{nk} - x|| \geq 1$, for all $x \in \langle x_1, \cdots, x_{n_{k-1}}\rangle$ and $z_{nj} \in \langle x_1, \cdots, x_{n_{k-1}}\rangle$, we have

$$||z_{nk} - z_{nj}|| \geq 1,$$

Contradiction holds. We conclude that the closed unit ball is not sequentially compact in an infinite dimensional normed space.Digressing a bit, let x be a point lying outside Y, a proper subspace of the normed

Notes

space X. A point in Y realizing the distance from x to Y is called the **best approximation** from x to Y. It always exists when Y is a finite dimensional subspace. However, things change dramatically when the subspace has infinite dimension. For instance, let Y be the closed subspace of $C[-1, 1]$ given by

$$\int_{-1}^{0} f(x)dx = 0, \quad \int_{0}^{1} f(x)dx = 0, \quad \forall f \in Y,$$

and h a continuous function satisfying

$$\int_{-1}^{0} h(x)dx = 1, \quad \int_{0}^{1} h(x)dx = -1.$$

One can show that the distance from h to Y is equal to 1, but it is not realized at any point on Y. You may try to prove this fact or consult chapter 5 of [L]. Later we will see that the best approximation problem has always a solution when the space X is reflexive.

CHECK YOUR PROGRESS

1.What is Banach space and define its properties.

2. Explain Sequential Compactness

3.4 ARZELA-ASCOLI THEOREM

From the last section, we know that not all bounded sequences in an infinite dimensional normed space have convergent subsequences. It is natural to ask what additional conditions are needed to ensure this property. For the space $C[a, b]$, a complete answer is provided by the Arzela-Ascoli theorem. This theorem gives a necessary and sufficient condition when a closed and bounded set in $C[a, b]$ is sequentially compact. In order to have wider applications, we shall work on a more general space $C(K)$, where K is a closed, bounded subset of Rn, instead of $C[a, b]$. As every continuous function in K attains its maximum and minimum, its sup-norm is always finite. It can be shown that $C(K)$ is a separable Banach space under the sup-norm.

The crux for sequential compactness for continuous functions lies on the notion of equi-continuity. Let E be a subset of $\mathbb{R}^n$. A subset F of $C(E)$ is **equi-continuous** if for every $\varepsilon > 0$, there exists some δ such that

$$|f(x) - f(y)| < \varepsilon, \text{ for all } f \in F, \text{ and } |x - y| < \delta,\ x, y \in E$$

Recall that a function is uniformly continuous in E if for each $\varepsilon > 0$, there exists some δ such that

$$|f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta,\ x, y \in E.$$

So, equicontinuity means that δ can further be chosen independent of individual functions in F.

There are various ways to show that a family of functions is equicontinuous. A function f defined in a subset E of $\mathbb{R}^n$ is called **Hölder continuous** if there exists some $\alpha \in (0, 1)$ such that

$$|f(x) - f(y)| \leq L|x - y|\alpha, \text{ for all } x, y \in E, \qquad (1)$$

for some constant L. The number α is called the Hölder exponent. The function is called **Lipschitz continuous** if (1) holds for α equals to 1. A family of functions F in $C(E)$ is said to satisfy a **uniform Hölder** or **Lipschitz condition** if all members in F are Hölder continuous with the same α or Lipschitz continuous and (1) holds for the same constant L. Clearly, such F is equi-continuous.

The following situation is commonly encountered in the study of differential equations. The philosophy is that equicontinuity can be obtained if there is a good, uniform control on the derivatives of functions in *F*.

Proposition 3.4.1. *Let F be a subset of C*(*G*) *where G is a convex set in* $\mathbb{R}^n$. *Suppose that each member in F is differentiable and there is a uniform bound on the partial derivatives of the functions in F. Then F is equicontinuous.*

Proof. For, *x* and *y* in *G*, $(1 - t)x + ty$, $t \in [0, 1]$, belongs to *G* by convexity. Let $\psi(t) \equiv f((1 - t)x + ty)$.
From the mean-value theorem

$\psi(1) - \psi(0) = \psi0(t^*)(1 - 0)$, $t* \in [0, 1]$,

and the chain rule

$$\psi'(t) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}((1-t)x + ty)(y_j - x_j),$$

$$|f(y) - f(x)| \le \sum_j \left|\frac{\partial f}{\partial x_j}\right||y_j - x_j| \le \sqrt{n}M|y - x|,$$

where $M = \sup\{|\partial f/\partial x_j(x)| : x \in G, j = 1, \ldots, n, f \in F\}$ after using Cauchy-Schwarz inequality. So *F* satisfies a uniform Lipschitz condition with the Lipschitz constant $n^{1/2} M$

Theorem 3.4.2. (**Arzela-Ascoli).** *Let F be a closed set in C*(*K*) *where K is a closed and bounded set in* $\mathbb{R}^n$. *Then F is sequentially compact if and only if it is bounded and equi-continuous.*

A set *E* is called bounded if there exists $M > 0$ such that $|f(x)| \le M$, for all $f \in E$ and $x \in K$.
In other words, it is a bounded set in the metric induced by the sup-norm. This theorem was proved for *C*[*a, b*] in the end of the nineteenth century by two italian mathematicians, the sufficient part by Ascoli and the necessary part by Arzela respectively.

We shall need the following useful fact.

Lemma 3.4.3. *Let E be a set in the metric space* (X, d). *Then*

(a) That E is sequentially compact implies that for any $\varepsilon > 0$, *there exist finitely many* ε*-balls covering E.*

(b) Assuming that E is closed and (X, d) *is complete, the converse of (a) is true.*

Proof. (a) Suppose on the contrary that there exists some $\varepsilon_0 >$ such that no finite collection of ε_0-balls covers E. For a fixed $x1$, the ball $B_{\varepsilon_0}(x1)$ does not cover E, so we can pick $x_2 \notin B_{\varepsilon_0}(x1)$. As $B_{\varepsilon_0}(x_1)$ and $B_{\varepsilon_0}(x_2)$ together do not cover E, there is $x3 \notin B_{\varepsilon_0}(x1) \cup B_{\varepsilon_0}(x_2)$. Continuing this way, we find a sequence $\{xn\}$ satisfying $xn \notin B\varepsilon0(x1) \cup \cdots \cup B\varepsilon0(xn-1)$. In particular, $d(xi, xj) \geq \varepsilon_0$ for distinct i, j, which shows that $\{x_n\}$ cannot have any convergent subsequence, a contradiction the sequential compactness of E.

(b). Let $\{xn\}$ be a sequence in E. We may assume that it has infinitely many distinct elements, otherwise the conclusion is trivial. Since E can be covered by finitely many balls of radius 1, there exists one, say $B1$, which contains infinitely many elements of E.

Next, as E can be covered by finitely many balls of radius 1/2, there exists $B2$ of radius 1/2 so that $B1 \cap B2$ contains infinitely many elements of E. Continuing this we get B_n of radius $1/n$ such that $B1 \cap B2 \cdots \cap Bn \cap E$ is non-empty for all n. Pick $x_{n_j} \in B1 \cap B2 \cdots \cap Bj \cap E$ with $nj-1 < n_j$. Then $\{x_{n_j}\}$ is a subsequence of $\{x_n\}$ which is also a Cauchy sequence. As X is complete and E is closed, it is convergent in E. We consider that E is sequentially compact.

We shall also use the following lemma from elementary analysis.

Lemma 3.4.4. *Let* $\{fn\}$ *be a bounded sequence of functions from the countable set* $\{z1, z2, \cdots, \}$ *to* F. *There is a subsequence of* $\{fn\}$, $\{gn\}$, *such that* $\{gn(zj)\}$ *is convergent for every* zj.

Proof. Since $\{f_n(z_1)\}$ is a bounded sequence in F, we can extract a subsequence $\{f_n^1\}$ such that $\{f_n^1\ (z1)\}$ is convergent. Next, as $\{f_n^1\}$ is bounded, it has a subsequence $\{f_n^2\}$ such that $\{f_n^2\ (z2)\}$ is convergent. Keep doing in this way, we obtain sequences $\{f_n^j\}$ satisfying

(i) $\{f_n^{j+1}\}$ is a subsequence of $\{f_n^j\}$ and

(ii) $\{f_n^j\ (z1)\}, \{f_n^j\ (z2)\}, \cdots, \{fn\ j(zj)\}$ are convergent. Then the diagonal sequence $\{gn\}$, $gn = f_n^n$, for all $n \geq 1$, is a subsequence of $\{fn\}$ which converges at every z_j.

The subsequence selected in this way is sometimes called to Cantor's diagonal sequence.

Proof of Arzela-Ascoli Theorem:

Assuming boundedness and equicontinuity of F, we would like to show that F is sequentially compact. Since K is sequentially compact in $\mathbb{R}^n$, by Lemma 3.3.3, for each $j \geq 1$, we can cover K by finitely many balls $D1/j(xj\ 1), \cdots, D1/j(xj\ K)$ where the number K depends on j. For any sequence $\{fn\}$ in F, by Lemma 3.3.4, we can pick a subsequence from $\{fn\}$, denoted by $\{gn\}$, such that $\{gn(x_k^j.)\}$ is convergent for each x_k^j. We claim that $\{gn\}$ is Cauchy in $C(K)$. For, due to the equi-continuity of F, for every $\varepsilon > 0$, there exists a δ such that $|gn(x) - gn(y)| < \varepsilon$, whenever $|x - y| < \delta$. Pick j_0, $1/j_0 < \delta$. Then for $x \in K$, there exists $xj0\ k$ such that

$$|x - x_k^{j_0}.\ | < 1/j_0 < \delta,$$

$$\begin{aligned} |g_n(x) - g_m(x)| &\leq |g_n(x) - g_n(x_k^{j_0})| + |g_n(x_k^{j_0}) - g_m(x_k^{j_0})| + |g_m(x_k^{j_0}) - g_m(x)| \\ &< \varepsilon + |g_n(x_k^{j_0}) - g_m(x_k^{j_0})| + \varepsilon. \end{aligned}$$

As $\{gn(x_k^{j_0}\}$ converges, there exists n_0 such that

$$|g_n(x_k^{j_0}) - g_m(x_k^{j_0})| < \varepsilon, \quad \text{for all } n, m \geq n_0. \qquad (2)$$

Here n_0 depends on $x_k^{j_0}$. As there are finitely many $xj\ k0$'s, we can choose some N_0 such that (2) holds for all $x_k^{j_0}$ and $n, m \geq N0$. It follows that $|gn(x) - gm(x)| < 3\varepsilon$, for all $n, m \geq N_0$, i.e., $\{gn\}$ is Cauchy in $C(K)$. By the completeness of $C(K)$ and the closedness of $\mathcal{F}$, $\{gn\}$ converges to some function in F.

Conversely, by Lemma 2.16, for each $\varepsilon > 0$, there exist $f_1, \cdots, f_N \in F$ such that $F \subset$ SN j=1 $B\varepsilon(fj)$ where N depends on ε. So for any $f \in F$, there exists fj such that

$$|f(x) - f_j(x)| < \varepsilon, \qquad \text{for all } x \in K.$$

As each f_j is continuous, there exists δj such that $|f_j(x) - f_j(y)| < \varepsilon$ whenever $|x - y| < \delta j$. Letting $\delta = \min\{\delta 1, \cdots, \delta N\}$, then

$$|f(x) - f(y)| \leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| < 3\varepsilon,$$

for $|x - y| < \delta$, so S is equicontinuous.

As F can be covered by finitely many 1-balls, it is also bounded.

We have completed the proof of Arzela-Ascoli theorem.

We note the following useful corollary of the theorem, sometimes called Ascoli's theorem.

Corollary 3.4.5. A sequence in C(K) where K is a closed, bounded set in Rn has a convergent subsequence if it is uniformly bounded and equicontinuous.

Proof. Let F be the closure of the sequence $\{fj\}$. As this sequence is uniformly bounded, there exists some M such that

$$|fj(x)| \leq M, \ \forall x \in K, j \geq 1.$$

Consequently, any limit point of $\{fj\}$ also satisfies this estimate, that is, F is bounded in $C(K)$. Similarly, by equicontinuity, for every $\varepsilon > 0$, there exists some δ such that

$$|f_j(x) - f_j(y)| < \varepsilon/2, \forall x, y \in K, \ |x - y| < \delta.$$

As a result, any limit point f of $\{f_j\}$ satisfies

$$|f(x) - f(y)| \leq \varepsilon 2 < \varepsilon, \qquad \forall x, y \in K, |x - y| < \delta,$$

so F is also equicontinuous. Now the conclusion follows from the Arzela-Ascoli theorem.

We present an application of Arzela-Ascoli theorem to ordinary differential equations. Consider the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(0) = 0,$$

where f is a continuous function defined in $[-a, a] \times [-b, b]$. We are asked to find a differentiable function $y(x)$ so that this equation is satisfied for x in some interval containing the origin. Under the further assumption that f satisfies the "Lipschitz condition": For some constant

$$L|f(x, y1) - f(x, y2)| \leq L|y1 - y2|, \text{ for all } x \in [-a, a], y1, y2 \in [-b, b],$$

we learn from a course on ordinary differential equations that there exists a *unique* solution to this initial value problem defined on the interval $I = (-a0, a0)$, where $a0 = \min\{a, b/M\}$, where

$$M = \max\{|f(x, y)| : (x, y) \in [-a, a] \times [-b, b]\}.$$

Now, let us show that the Lipchitz condition can be removed as far as *existence* is in concern. First of all, by Weierstrass approximation theorem, there exists a sequence of polynomials $\{fn\}$ approaching f in $C([-a, a] \times [-b, b])$ uniformly. In particular, it means that $Mn \to M$, where

$$M_n = \max\{|fn(x, y)| : (x, y) \in [-a, a] \times [-b, b].$$

As each fn satisfies the Lipschitz condition (why?), there is a unique solution yn defined on $I_n = (-a_n, a_n)$, $a_n = \min\{a, b/M_n\}$ for the initial value problem

$$\frac{dy}{dx} = f_n(x, y), \quad y(0) = 0.$$

Notes

From $|dy_n/dx| \le Mn$ and $\lim_{n\to\infty} Mn = M$, we know from Proposition 2.14 that *{yn}* forms an
equicontinuous family.

Clearly, it is also bounded. By Arzela-Ascoli theorem, it contains a subsequence $\{y_n^j\}$ converging uniformly to a continuous function $y \in I$ on every subinterval $[\alpha, \beta]$ of I and $y(0) = 0$ holds. It remains to check that y solves the differential equation for f.

Indeed, each y_n satisfies the integral equation

$$y_n(x) = \int_0^x f(t, y_n(t))dt, \quad x \in I_n.$$

As $\{y_n^j\} \to y$ uniformly, $\{f(x, y_n^j(x))\}$ also tends to $f(x, y(x))$ uniformly. By passing to limit in the formula above, we conclude that

$$y(x) = \int_0^x f(t, y(t))dt, \quad x \in I$$

holds. By the fundamental theorem of calculus, y is differentiable and a solution to our initial value problem.
The solution may not be unique without the Lipchitz condition. Indeed, the function $y_1(x) \equiv 0$
solves the initial value problem $y'(x) = y^{1/2}$, $y(0) = 0$, and yet there is another solution given by $y_2(x) = x^2/4$, $x \ge 0$ and vanishes for $x < 0$.

CHECK YOUR PROGRESS

3. Define - **Hölder continuous & Lipschitz continuous**

4. State **Arzela-Ascoli Theorem**

3.5 LET'S SUM UP

We concluded the application of Weierstrass approximation theorem in Arzela-Ascoli theorem. We comprehended Arzela-Ascoli theorem. The Arzela-Ascoli Theorem is a very important technical result, used in many branches of mathematics. Aside from its numerous applications to Partial Differential Equations, the Arzela-Ascoli Theorem is also used as a tool in obtaining Functional Analysis results, such as the compactness for duals of compact operators

3.6 KEYWORDS

Translational invariance implies that, at least in one direction, the object is infinite: for any given point p, the set of points with the same properties due to the **translational symmetry** form the infinite discrete set $\{p + na \mid n \in Z\} = p + Z a$

Vector: A **vector** is an object that has both a magnitude and a direction. Geometrically, we can picture a **vector** as a directed line segment, whose length is the magnitude of the **vector** and with an arrow indicating the direction.

Completeness- every nonempty set that has an upper bound has a smallest such bound, a property not possessed by the rational numbers.

3.7 QUESTION FOR REVIEW

1. Show that a metric d induced from a norm on the vector space X satisfies (i) $d(x+z, y+z) = d(x, y)$
and (ii) $d(\alpha x, \alpha y)) = |\alpha| d(x, y)$. Use these properties to find two examples of metrics (on vector
spaces) which are not induced by norms.

2. (a) Prove that ℓ^1 a proper vector subspace of $\ell^P$for $p > 1$.
(b) Now both the 1-norm and p-norm are norms on ℓ^1. Are they equivalent?

3. Let $|| \cdot ||_1$ and $|| \cdot ||_2$ be two norms on the vector space X.

(a) Show that $||x||_M = \max\{||x||_1, ||x||_2\}$ is again a norm on X.

(b) Is this true for $||x||_m = \min\{||x||_1, ||x||_2\}$?

4. (a) Establish the estimates

$$||x||_\infty \leq ||x||_p \leq n^{1/p} \, ||x||_\infty, \; p \geq 1.$$

It shows that all p-norms are equivalent on Rn.

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3.9 ANSWER TO CHECK YOUR PROGRESS

1. Refer explanation of step 3 – 3.1.1

2. Refer explanation – 3.2

3. Provide definition – 3.3

4. Provide statement and proof -3.3.2

UNIT 4: DUAL SPACE I

STRUCTURE

4.0 OBJECTIVE

Understand the concept of Linear Functionals

Enumerate the Concrete Dual Spaces

Comprehend Hahn-Banach Theorem

4.1 INTRODUCTION

In this chapter we further our study of Banach spaces by examining continuous linear functional on them. Each of these functional gives very limited information on the space, but as a whole they become enormously helpful. The fundamental Hahn-Banach theorem guarantees there are sufficiently many such functional for various purposes. They form a normed space called the dual space of the original space. We identify the dual spaces of $\mathbb{F}^n$, $\ell^p, 1 \leq p < \infty$, and $C[a, b]$. Reflexive spaces arise naturally when we study the dual of dual spaces.

4.2 LINEAR FUNCTIONALS

Any linear function from a vector space X to its scalar field $\mathbb{F}$ is called a **linear functional**. It is clear that the collection of all linear functional from X to $\mathbb{F}$, denoted by $L(X, \mathbb{F})$, forms a vector space over F under point wise addition and scalar multiplication of functions. Linear functional play a crucial role in the study of the structure of vector spaces. There are two subspaces associated to a linear functional, namely, its image and its kernel, and the latter is more relevant. Indeed, let $\Lambda \in L(X, \mathbb{F})$, the **null space** (or **kernel**) of Λ is given by $N(\Lambda)$ (or ker Λ) the set $\{x \in X : \Lambda x = 0\}$. It is clear that the kernel $N(\Lambda)$ forms a subspace of X and it is proper if and only if Λ is not identically zero.

Proposition 4.2.1. *Let X be a vector space over* $\mathbb{F}$. *Then*

(a) L(X, F) is a vector space over F,

(b) $N(\Lambda)$ is a subspace of X for any $\Lambda \in L(X, F)$, and

(c) if Λ is non-zero, then for any $x_0 \in X \setminus N(\Lambda)$, $X = N(\Lambda) \oplus \langle x_0 \rangle$.

Proof. (a) and (b) can be verified directly. It suffices to prove (c). Let $x0$ be a point satisfying $\Lambda x_0 \neq 0$.

For any $x \in X$, the vector $y = x - \lambda x_0$ where $\lambda = \Lambda x/\Lambda x_0$ belongs to $N(\Lambda)$: $\Lambda(x - \lambda x_0) = \Lambda x - \lambda \Lambda x_0 = 0$.

Therefore, $x = y + \lambda x_0$, that is, $X = N(\Lambda) + \langle x_0 \rangle$. To show this is a direct sum, suppose that $x =$

$y1 + \lambda x_0 = y2 + \mu x_0$. Then $y1 - y2 = (\mu - \lambda)x_0$, so $(\mu - \lambda)\Lambda x_0 = \Lambda(y1 - y2) = 0$ implies that $\mu = \lambda$ and $y1 = y2$.

The meaning of (c) can be understood better by looking at the finite dimensional situation. Any

linear functional Λ on Fn is completely determined by its values at a basis. For instance, consider the canonical basis $e1, \cdots, en$ and let $\alpha_j = \Lambda e_j$, $j = 1, \cdots, n$, for $x \in \mathbb{F}^n$, $x = \sum_1^n x_j e_j$. Then

$$\Lambda x = \Lambda\left(\sum_1^n x_j e_j\right) = \sum_1^n \alpha_j x_j = \alpha_1 x_1 + \cdots + \alpha_n x_n,$$

Gives the general formula for a linear functional on $\mathbb{F}^n$. What is $N(\Lambda)$ for a nonzero Λ? Apparently it is composed of the set $\{x \in \mathbb{F}^N : \alpha_1 x_1 + \cdots + \alpha_n x_n = 0\}$. When $\mathbb{F}^n = \mathbb{R}^n$, this is precisely the equation for a hyperplane passing through the origin whose normal direction is given by $(\alpha 1, \cdots, \alpha n)/(\alpha_1^2 + \cdots + \alpha_n^2)^{1/2}$. In general, a hyperplane is one dimension lower than its ambient space. Thus (c) tells us that in infinite dimensional situation this is still true: After adjoining a single dimension (spanned by $x0$) to it, $N(\Lambda) \oplus \langle x_0 \rangle$ is the entire space.

The abundance of linear functionals can be seen by the following abstract consideration. Let B be a Hamel basis for X. For each x in this basis we define a functional Λ_x by setting $\Lambda_x(\alpha x) = \alpha$ and $\Lambda_x y = 0$ for any y in B distinct from x where α is a fixed scalar. As every vector can be written as a finite linear combination of elements from B, it is easy to see that Λ_x extends to become a linear functional on X.

Moreover, one readily verifies that all these Λx's form a linearly independent set, so $L(X, F)$ is of infinite dimension.

When it comes to a normed space $(X, ||\cdot||)$, a linear functional may not be related to the norm

structure of the space. To have good interaction with the norm structure it is more desirable to look at linear functionals which are also continuous with respect to the norm. By linearity it is easy to show that a continuous linear functional is continuous everywhere once it is so at a single point. A related notion of continuity of a linear functional is its boundedness. We used to call a function bounded if its image is a bounded set. For linear functionals boundedness has a different meaning. We call a linear functional Λ **bounded** if it maps any bounded set in X to a bounded set in $\mathbb{F}$.

In other words, for any bounded S in X, there exists a constant C such that $|\Lambda x| \le C$, for all $x \in S$. By linearity for Λ to be bounded it suffices that it maps a ball to a bounded set in the scalar field, or equivalently in the form of an estimate, there exists a constant C' such that $|\Lambda x| \le C'||x||$ for all $x \in X$. It turns out that for a linear functional continuity and boundedness are equivalent. We put all these in the following proposition.

Notes

Proposition 4.2.2. *Let* $\Lambda \in L(X, \mathbb{F})$ *where* X *is a normed space. We have*

(a) Λ *is continuous if and only if* Λ *is continuous at one point.*
(b) Λ *is bounded if and only if there exists* $C > 0$ *such that* $|\Lambda x| \leq Ckxk$ *for all* x.
(c) Λ *is continuous if and only if* Λ *is bounded.*

Proof. (a) It suffices to show the "if" part. Suppose Λ is continuous at $x0$, that's, $\Lambda x_n \to \Lambda x_0$ whenever $x_n \to x_0$ in X. For any $x1 \in X$ and $x_n \to x_1$, we have

$$x_n - x_1 + x_0 \to x_0,$$

so

$$\Lambda(x_n - x_1 + x_0) \to \Lambda(x_0).$$

By linearity,
$\Lambda xn - \Lambda x_1 + \Lambda x_0 \to \Lambda x_0$ which means $\Lambda x_n \to \Lambda x_1$.

(b) Let Λ be bounded and take S to be the closed unit ball $B1(0)$. Then we can find a constant C_1
such that $|\Lambda x| \leq C_1$. For any nonzero x, $x/ ||x|| \in B_1(0)$, we have $|\Lambda(x/||x||)| \leq C_1$, i.e., $|\Lambda x| \leq C_1||x||$.
Conversely, let S be a bounded set. We can find a large ball $B_R(0)$ to contain S. Then for any x in S, $|\Lambda x| \leq C||x|| \leq C_R$.

(c) If Λ is not bounded, there exists $kxnk \leq M$ but $|\Lambda xn| \to \infty$. Then the sequence $\{yn\}$, $y_n = x_n/|\Lambda x_n|$, satisfies $||y_n|| \to 0$ but $|\Lambda y_n| = 1$, so Λ cannot be continuous.
On the other hand, let $x_n \to x_0$, that's, $||x_n - x_0|| \to 0$. When Λ is bounded, by (b) $|\Lambda x_n - \Lambda x_0| = |\Lambda(xn - x0)| \leq Ckxn - x0k \to 0$, so Λ is continuous.
Proposition 4.2.3. A linear functional on a normed space is bounded if and only if its kernel is closed.
We use X' to denote all bounded linear functionals on X. It is clear that X' is a subspace of $L(X, F)$. When X is of finite dimension, we have seen that every linear functional is of the form

$$\Lambda x = \sum_{1}^{n} \alpha_j x_j,$$

hence it is continuous. Thus *X'* = *L*(*X*, 𝔽) when *X* is of finite dimension. However, this is no longer true when *X* is of infinite dimension. Let *B* be a Hamel basis for an infinite dimensional space *X*. We may pick a countably infinite set {*x*1, *x*2, *x*3, · · · }, $||x_k|| = 1$, ∀*k*, from *B* and define *T* by assigning $T x_k = k$, *k* = 1, 2, · · · and *T x* = 0 for the remaining vectors in *B*. As *B* is a basis, *T* can be extended to become a linear functional on *X*. Clearly it cannot be bounded.

Now we come to the norm structure on *X'* inherited from *X*

Proposition 4.2.4. Let X be a normed space and Λ a bounded linear functional on X and define

$$||\Lambda|| = \sup_{x \neq 0} \frac{|\Lambda x|}{||x||}.$$

Then || · || *is a norm on X'.*

Before the proof of this proposition we point out a few things. First, the operator norm *k*Λ*k* is also given by

$$\|\Lambda\| = \sup_{\|x\|=1} |\Lambda x|.$$

Or

$$\|\Lambda\| = \sup_{\|x\|\leq 1} |\Lambda x|.$$

Second, we always have the useful inequality

$$|\Lambda x| \leq \|\Lambda\| \|x\|, \quad \text{for all } x \in X.$$

Third, the definition of the operator norm is basically the sup-norm for continuous functions. However, as the supermom is always infinity for any nonzero linear functional, we modify it by taking the supremum over the unit ball {||*x*|| ≤ 1}. Thanks to boundedness of the functional this supremum is always a finite number, and thanks to linearity it satisfies (N1).

Proof. Clearly (N1) and (N2) hold. To verify (N3), for $\Lambda 1$, $\Lambda 2 \in X0$ and $x \in X$, $||x|| = 1$,

$$|(\Lambda_1 + \Lambda_2)(x)| = |\Lambda_1 x + \Lambda_2 x| \leq |\Lambda_1 x| + |\Lambda_2 x| \leq \|\Lambda_1\| + \|\Lambda_2\|.$$

$$\|\Lambda_1 + \Lambda_2\| = \sup_{\|x\|=1} |(\Lambda_1 + \Lambda_2)(x)| \leq \|\Lambda_1\| + \|\Lambda_2\|.$$

From Proposition 4.1.4, (X', $|| \cdot ||$) forms a normed space called the **dual space** of (X, $|| \cdot ||$). The norm on X' is called the **operator norm** sometimes. It is a bit surprising that X' behaves better than X as implicated by the following proposition.

Proposition 4.2.5. *The dual space X' of a normed space X is a Banach space.*

Proof. Let $\{\Lambda k\}$ be a Cauchy sequence in X', that's, for every $\varepsilon > 0$, there is some k_0 such that $||\Lambda_k - \Lambda_l|| < \varepsilon$ for all $k, l \geq k_0$. For each $x \in X$,

$$|\Lambda_k x - \Lambda_l x| \leq \|\Lambda_k - \Lambda_l\| \|x\| \leq \varepsilon \|x\|, \qquad (1)$$

which shows that $\{\Lambda kx\}$ is a Cauchy sequence in F. By the completeness of F, $\lim_{k\to\infty} \Lambda kx$ exists for every $x \in X$. Setting $\Lambda x \equiv \lim_{k\to\infty} \Lambda_k x$, it is routine to check that Λ is linear. Moreover, by letting $l \to \infty$ in (1), we have

$$|\Lambda_k x - \Lambda x| \leq \varepsilon \|x\|, \ k \geq k_0. \qquad (2)$$

It follows that

$$|\Lambda x| \leq |\Lambda_{k_0} x - \Lambda x| + |\Lambda_{k_0} x| \leq (\varepsilon + \|\Lambda_{k_0}\|)\|x\|,$$

so Λ is also bounded. From (2) we have

$$|\Lambda_k x - \Lambda x| \leq \varepsilon, \qquad k \geq k_0,$$

for all x, $||x|| = 1$, so $\Lambda k \to \Lambda$ in operator norm.

4.3 CONCRETE DUAL SPACES

We determine the dual spaces of $\mathbb{F}^n$ and ℓ^p, $1 \le p < \infty$ in this section. Recall that we identify two normed spaces $(X_1, ||\cdot||_1)$ and $(X_2, ||\cdot||_2)$ if there exists a norm-preserving linear isomorphism from X_1 to X_2.

Proposition 4.3.1. For $1 \le p < \infty$, the dual space of ℓ^p is given by ℓ^q where p and q are conjugate.

Proof. We only consider $p > 1$ and leave the case $p = 1$ to you.

We define a map from $(\ell^p)0$ to ℓ^q as follows. For each $\Lambda \in (\ell^p)0$, let $\Phi\Lambda$ be the sequence $\alpha =$
$(\Lambda e1, \Lambda e2, \cdots)$ where $\{ej\}$ is the "canonical sequence". This is a linear map from $(\ell^p)0$ to the space of sequences. We claim that its image belongs to ℓ^q.

Letting $\alpha^N = (\text{sgn}(\alpha_1)|\alpha_1|^{q-1}, \cdots, \text{sgn}(\alpha_N)|\alpha_N|^{q-1}, 0, 0, \cdots)$ and using the inequality $|\Lambda x| \le ||\Lambda||||x||_p$ for $x = \alpha^N$ we have, as

$$|\Lambda\alpha^N| = \sum_1^N |\alpha_j|^q \text{ and } \|\alpha^N\|_p = \left(\sum_1^N |\alpha_j|^q\right)^{\frac{1}{p}},$$

$$\left(\sum_1^N |\alpha_j|^q\right)^{\frac{1}{q}} \le \|\Lambda\|.$$

Letting $N \to \infty$, we have

$$\|\Phi(\Lambda)\|_q = \|\alpha\|_q \le \|\Lambda\|. \quad (3)$$

We have shown that Φ maps $(\ell^p)'$ into ℓ^q.

To show that Φ is onto we construct its inverse. For each α in `q we define $\Psi\alpha = \Lambda\ \alpha$ where $\Lambda\alpha$ is given by $\Lambda\alpha x =$ P$j\ \alpha jxj$. By Hölder inequality, this map is well-defined and

$$|\Lambda_\alpha x| \leq \|\alpha\|_q \|x\|_p, \quad \text{for all } x \in \ell^p.$$

$$\|\Psi\alpha\| = \|\Lambda_\alpha\| \leq \|\alpha\|_q. \tag{4}$$

$$\Phi\Psi\alpha = \alpha, \quad \forall\alpha \in \ell^q.$$

It follows that

Next we claim that

Indeed, for $\alpha \in \ell^q$, $\Phi\Psi\alpha = ((\Psi\alpha)e_1, (\Psi\alpha)e_2, \cdots) = (\alpha 1, \alpha 2, \cdots) = \alpha$, so the claim holds. This claim shows in particular that Φ is onto. By combining it with (3) and (4), we have

$$||\Lambda|| = ||\Phi\Psi\Lambda|| \leq ||\Phi\Lambda||_q \leq ||\Lambda||,$$

whence $||\Phi\Lambda||_q = ||\Lambda||$, that is, Φ is norm-preserving. We have succeeded in constructing a norm-preserving linear isomorphism from $(\ell^p)'$ to ℓ^q, so these two spaces are the same.

A similar but simpler proof shows that the dual of $\mathbb{F}^n$ under the $p-$ norm is itself under the q-norm for $p \in [1, \infty]$.

It is a standard result in real analysis that the dual space of $L^p(a, b)$, $1 \leq p < \infty$, is $L^q(a, b)$ where

q is conjugate top. But it is not true when p is infinity.

CHECK YOUR PROGRESS

1. Explain Dual space and operator norm

2. Enumerate - The dual space X' of a normed space X is a Banach space.

4.4 HAHN-BANACH THEOREM

In the last section we identified the dual space of F*n* and `*p*, $1 \leq p < \infty$. In particular, it shows that there are many non-trivial bounded linear functional in these spaces. However, in a general normed space it is not clear how to find even one which distinguishes two points. The theorem of Hahn-Banach ensures that we can always do this. This extremely useful theorem, which is formulated as a statement on extension, is one of the most fundamental results in functional analysis.

Considering its applications in later chapters, it is necessary to formulate the theorem not only in the setting of a normed space but in a vector space. We call a function *p* defined in a vector space *X* to$(-\infty, \ \infty]$ **sub additive** if for all *x*, *y* in *X*,

$$p(x + y) \leq p(x) + p(y),$$

and **positive homogeneous** if for all *x* in *X* and $\alpha \geq 0$,

$$p(\alpha x) = \alpha p(x).$$

Note that the norm is a subadditive, positive homogenous function due to (N2) and (N3). A nonnegative, sub additive, positive homogeneous function on a vector space is sometimes called a **gauge** or a **Murkowski functional**.

Any positive multiple of the norm is a gauge on a normed space. Other gauges can be found as follows. Let *C* be a non-empty convex set containing 0 in a vector space *X*. Define

$$p_C(x) = \inf\{\alpha : x \in \alpha C, \ \alpha > 0\}$$

and set $p_C(x) = \infty$ if no such α exists. We claim that *pC* is a gauge. Clearly, $pC(\alpha x) = \alpha pC(x)$ for every positive α. On the other hand, consider *x*, *y* in *X* where *pC*(*x*) and *pC*(*y*) are finite (subadditivity holds trivially if they are not). According to the definition of a gauge, for every $\varepsilon > 0$, there exist positive α, β satisfying $pC(x) \geq \alpha - \varepsilon$, $pC(y) \geq \beta - \varepsilon$ and $x/\alpha, \ y/\beta \in C$. Therefore, by the convexity of *C*

$$\frac{x+y}{\alpha+\beta} = \frac{\alpha}{\alpha+\beta}\frac{x}{\alpha} + \frac{\beta}{\alpha+\beta}\frac{y}{\beta} \in C,$$

Notes

which implies that

$$p_C(x + y) \le \alpha + \beta \le p_C(x) + p_C(y) + 2\varepsilon.$$

So $p_{ct.}$ is sub additive. Gauges of convex sets form an important class of sub additive, positive homogeneous functions.

Now we state and prove a general form of the Hahn-Banach theorem.

Theorem 4.4.1. Let X be a vector space and p a sub additive, positive homogeneous function in X. Suppose $\Lambda \in L(Y, F)$ where Y is a proper subspace of X satisfies

$$R_e\Lambda x \le p(x), \qquad \textit{for all } x \in Y.$$

Then there is an extension of Λ to L(X, F), Λ e, such that

$$R_e\widetilde{\Lambda}\,\mathrm{e}x \le p(x), \qquad \textit{for all } x \in X.$$

We will treat the real case first. The complex case can be deduced from the real case. The technical part of the proof of this theorem is contained in the following lemma.

Lemma 4.4.2 (One-Step Extension). *Let* F = R, $\Lambda \in L(Y, \mathrm{R})$ *and* $x_0 \in X \setminus Y$. There exists an

extension Λ1 of Λ on $\langle Y, x_0\rangle$ such that $\Lambda 1x \le p(x)$.

Proof. Let $Y_1 = \langle Y, x_0\rangle$ Every element in Y_1 is of the form $x = y + c\,x_0$, $y \in Y$, $c \in \mathrm{R}$. Any linear

functional Λ1 extending Λ satisfies

$$\Lambda_1 x = \Lambda_1(y + c\,x_0) = \Lambda_1 y + c\Lambda_1\,x_0 = \Lambda y + c\Lambda_1\,x_0.$$

Conversely, by assignment any value to $\Lambda 1x_0$ one obtains an extension of Λ in this way. Nevertheless, the point is to determine the value of $\Lambda 1\,x_0$ so that $\Lambda 1x \le p(x)$ on $Y1$. To show such choice is possible, let's focus at $c = \pm 1$. For an admissible extension, one should have

$$\Lambda_y \pm \Lambda_1 x_0 \le p(y \pm x_0), \qquad (5)$$

or,

$$\Lambda_1 x_0 \leq p(y + x_0) - \Lambda y$$

And

$$\Lambda_y - p(y - x_0) \leq \Lambda 1 x_0.$$

It implies that for all $y, z \in Y$,

$$\Lambda_z - p(z - x_0) \leq \Lambda 1 x_0 \leq p(y + x_0) - \Lambda_y.$$

Therefore, if

$$\alpha \equiv \sup_{z \in Y}(\Lambda z - p(z - x_0)) \leq \beta \equiv \inf_{y \in Y}(p(y + x_0) - \Lambda y) \qquad (6)$$

holds, we can pick any $\gamma \in [\alpha, \beta]$ and set $\Lambda 1 x0 = \gamma$, so that (3.5) holds. Before verifying this, let's show that it implies $\Lambda 1$ is our desired extension. In fact, by linearity, for any $c > 0$,

$$\begin{aligned} \Lambda_1(y \pm cx_0) &= \Lambda y \pm c\Lambda_1 x_0 = c(\Lambda(\frac{y}{c}) \pm \Lambda_1 x_0) \\ &\leq cp(\frac{y}{c} \pm x_0) \\ &= p(y \pm cx_0), \end{aligned}$$

so $\Lambda_1 x \leq p(x)$ for all x in $Y1$. It remains to verify (6). But this is easy. We write (6) as

$$\Lambda z - p(z - x0) \leq p(y + x0) - \Lambda y, \ \forall y, z \in Y,$$

and it holds if and only if

$\Lambda(y + z) \leq p(z - x0) + p(y + x0)$.

Certainly this is true by the subaddivity of p

$\Lambda(y + z) \leq p(y + z) = p(y + x0 - x0 + z) \leq p(y + x0) + p(z - x0)$.

Proof of Theorem 4.4.1. Let $\mathbb{F} = \mathbb{R}$ first. Set $\mathcal{D} = \{(Z, T): Z$ is a subspace of X containing Y and

$T \in L(Z, \mathbb{R})$ is an extension of Λ satisfying $Tx \leq p(x)$ on $Z.\}$. D is non-empty since $(Y, \Lambda) \in \mathcal{D}$. A

relation "$\leq$" is defined on $\mathcal{D}$, $(Z_1, T_1) \leq (Z_2, T_2)$ if and only if (a) Z_1 is a subspace of Z_2 and (b) T_2 extends $T1$. We check easily that $(\mathcal{D}, \leq)$ is a poset.

We claim that each chain $\mathcal{C}$ in $(\mathcal{D}, \leq)$ has an upper bound. Indeed, for all Z in $\mathcal{C}$, let

$$Z^* = \bigcup_{Z_\alpha \in \mathscr{C}} Z_\alpha \text{ and } T^* z = T_\alpha z \text{ for } z \in Z_\alpha.$$

We show that Z^* is a subspace of X. Let $z_1, z_2 \in Z^*$. Then $z_1 \in Z\alpha$ and $z_2 \in Z_\beta$ for some α, β. As C is a chain, either $Z\alpha$ is a subspace of Z_β or the other way around. Let's assume the latter, so $z_1, z_2 \in Z\alpha$ and $\lambda z_1 + \mu z_2 \in Z\alpha \subset Z^*$. Z^* is a subspace. By a similar reason we can show that $T\alpha z = T\beta z$ if $z \in Z\alpha \cap Z\beta$, so T^* is well-defined. For any $z \in Z^*$, there exists some Z_α containing z, so $T^* z = T\alpha z \leq p(z)$.

We have shown that (Z^*, T^*) is an upper bound for C . Now we apply Zorn's lemma to conclude that there is a maximal element (Z_{max}, T_{max}) in D. We claim that $Z_{max} = X$. For, if this is not true, we can find $x_0 \in X \setminus Zmax$. Using the one-step extension lemma, we find $T1$ on $Z1 = \langle Z_{max}, x_0 \rangle$extending T_{max} and $T_1 x \leq p(x)$, $x \in Z1$. So $(Z1, T1) \in D$, that is to say, (Z_{max}, T_{max}) cannot be a maximal element. This contradiction shows that $Z_{max} = X$, and Λ e $\equiv T_{max}$ is our desired extension of Λ. This completes the proof of the general Hahn-Banach theorem
for the real case.

To treat the complex case, we need the following lemma. It asserts that any complex linear functional is uniquely determined by its real or imaginary part.

Lemma 4.4.3. (a) Let Λ be in L(X, C) where X is a complex vector space. Then its real and imaginary parts are in L(X, R) when X is regarded as a real vector space. Furthermore,

$$\Lambda x = Re\Lambda x - iRe\Lambda(ix), \qquad \textit{for all } x \in X. \quad (7)$$

(b) Conversely, for any Λ_1 in L(X, R), there exists a unique element in L(X, C) taking Λ_1 as its real part so that the above formula holds.

Proof. It is clear that both the real and imaginary parts of a complex linear functional are linear functionals over the reals. Write

$$\Lambda x = Re(\Lambda x) + i\, Im(\Lambda x) \equiv \Lambda_r x + i\, \Lambda i x.$$

We claim that they are related by

$$\Lambda_r(ix) = -\Lambda i(x),\ \Lambda i(ix) = \Lambda r x, \qquad (8)$$

so (7) holds. To see this, simply use the linearity of Λ over C to get

$$\Lambda(ix) = i\, \Lambda(x),$$

so,

$$\Lambda_r(ix) + i\, \Lambda i(ix) = i\, \Lambda r x - \Lambda i x,$$

and (8) holds.

Now, given a real linear functional Λ_1 on X, define

$$\Lambda x = \Lambda_1 x - i\Lambda(ix).$$

It is clear that the real part of Λ is equal to Λ1. It remains to check that Λ is linear. For $x1$, $x2 \in X$, and $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned} \Lambda(x_1 + x_2) &= \Lambda_1(x_1 + x_2) - i\Lambda_1(i(x_1 + x_2)) \\ &= \Lambda_1 x_1 - i\, \Lambda_1(i\, x_1) + \Lambda_1 x2 - i\, \Lambda_1(i\, x_2) \\ &= \Lambda x_1 + \Lambda x_2; \end{aligned}$$

$$\begin{aligned} \Lambda((\alpha + i\beta)x) &= \Lambda(\alpha x + i\beta x) = \Lambda(\alpha x) + \Lambda(\beta i x) \\ &= \Lambda_1(\alpha x) - i\Lambda_1(\alpha i x) + \Lambda_1(\beta i x) - i\Lambda_1(i\beta i x) \\ &= \alpha\Lambda_1 x - i\alpha\Lambda_1(ix) + \beta\Lambda_1(ix) + i\beta\Lambda_1 x \\ &= (\alpha + i\beta)\Lambda x; \end{aligned}$$

We complete the proof of the general Hahn-Banach theorem as follows. We first obtain a real extension Λ1 of the real part of the complex linear functional Λ satisfying $\Lambda 1x \le p(x)$ on X. By the lemma above, we find a complex linear functional Λ on ˜ X whose real part is given by Λ1 extending Λ. $\tilde{\Lambda}$ is our desired extension.

We have the following version of Hahn-Banach theorem on normed spaces.

Notes

Theorem 4.4.4. *Let* $(X, ||\cdot||)$ *be a normed space and Y a proper subspace of X. Then any* $\Lambda \in Y0$ *admits an extension to some* Λ e $\in X0$ *with* $||\Lambda$ e$||$ = $||\Lambda||$.

Notice that from definition, in general, we have

$$||\widetilde{\Lambda}|| = \sup_{x \in X \setminus \{0\}} \frac{|\widetilde{\Lambda}x|}{||x||} \geq \sup_{x \in Y \setminus \{0\}} \frac{|\Lambda x|}{||x||} = ||\Lambda||.$$

It suffices to establish the inequality from the other direction.

Proof. Consider first the real case. Taking $p(x) = ||\Lambda||||x||$, we apply the general Hahn-Banach theorem to obtain an extension of Λ, Λ, which satisfies$\tilde{\Lambda}$, $\tilde{x} \leq ||\Lambda||||x||$. Replacing x by $-x$, we get $-\Lambda\ \tilde{x} \leq ||\Lambda||||-x|| = ||\Lambda||||x||$, so $|\tilde{\Lambda}\ x| \leq ||\Lambda||||x||$, *which* implies $||\tilde{\Lambda}|| \leq ||\Lambda||$ on X.

For the complex case, let Λ be an extension of Λ satisfying $Re\tilde{\Lambda}x \leq ||\Lambda||||x||$ on X. Replacing x by $-x$, we have $|Re\tilde{\Lambda}x| \leq ||\Lambda||||x||$. For any x , there is a complex number $e^{i\theta}$ such that $\tilde{\Lambda}\ x = |\tilde{\Lambda}\ x\ |e^{i\theta}$. It follows that $|\tilde{\Lambda}\ x| = \tilde{\Lambda}\ (e^{-i\theta}\ x) = Re\tilde{\Lambda}\ (\ e^{-i\theta x}) \leq ||\Lambda||||\ e^{-i\theta}\ x\ || = ||\Lambda||||x||$, that is, $||\tilde{\Lambda}|| \leq ||\Lambda||$.

The proof is completed.

CHECK YOUR PROGRESS

3. Define **sub additive and positive homogeneous**

4. **State** Hahn-Banach theorem on normed spaces

4.5 LET'S SUM UP

We have clarified the nonnegative, sub additive, positive homogeneous function on a vector space is sometimes called a **gauge** or a **Minkowski**

functional. Gauges of convex sets form an important class of subadditive, positive homogeneous functions. It turns out that for a linear functional continuity and boundedness are equivalent.

4.6 KEYWORDS

1. Extension - The **extension** of an object in abstract algebra, such as a group, is the underlying set of the object. The **extension** of a set is the set itself.
2. Suffice - In mathematics, a condition that must be satisfied for a statement to be true and without which the statement cannot be true.
3. Notions - are called types if each object belongs to only one of them, which is then also called the type of the variables that can name it.

4.7 QUESTION FOR REVIEW

1. Show that a linear functional Λ on a normed space is bounded if and only if its kernel is closed

2. Let $(X, ||\cdot||))$ be an infinite dimensional normed space. Given any $\Lambda 1$, $\cdots$, Λn in X'. Show that there exists a non-zero point $x \in X$ satisfying $\Lambda jx = 0$ for all $j = 1, \cdots, n$.

3. Show that $(\ell^1)' = \ell^\infty$

4.8 SUGGESTED READINGS

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4.9 ANSWER TO CHECK YOUR PROGRESS

1. Support explanation with proposition & proof – 4.1.4

2. Provide proof – 4.1.5

3. Provide explanation – 4.3

4. Provide statement and proof -4.3.4

UNIT 5: DUAL SPACES II

STRUCTURE

5.0 OBJECTIVE

Understand the concept of Consequences of Hahn-Banach Theorem

Comprehend The Dual Space of Continuous Functions

Enumerate the concept of Reflexive Spaces

5.1 INTRODUCTION

In the branch of functional analysis, a dual space refers to the space of all continuous linear functional on a real or complex Banach space. The dual space of a Banach space is again a Banach space when it is endowed with the operator norm.

5.2 CONSEQUENCES OF HAHN-BANACH THEOREM

Theorem 5.2.1. *Let* $(X, ||\cdot||)$ *be a normed space and* Y *a closed subspace of* X*. For any* $x_0 \in X \setminus Y$*, there exists* $\Lambda \in X'$, $||\Lambda|| = 1$, *satisfying*

$$\Lambda x_0 = dist(x0, Y),$$

and

$$\Lambda y = 0, \text{ for all } y \in Y.$$

Proof. Let d = dist(x_0, Y). It is positive because Y is closed and x0 stays outside Y . In the subspace Y1 =⟨Y, x_0⟩, every vector can be written uniquely in the form y + αx_0. We define Λ_0 on Y1 by setting

$$\Lambda_0(y + \alpha x0) = \alpha\|x_0\|.$$

Then Λ_0 is linear and vanishes on Y. Moreover, using

$$0 < d = \inf_{z \in Y} \|x_0 + z\| \leq \frac{1}{|\alpha|}\|\alpha x_0 + y\|, \ \forall y \in Y,$$

$$|\Lambda_0(y + \alpha x_0)| \leq |\alpha|\|x_0\| \leq \frac{\|x_0\|}{d}\|y + \alpha x_0\|,$$

We have

in other words, $\Lambda_0 \in Y_1'$ and

$$\|\Lambda_0\| \leq \frac{\|x_0\|}{d}.$$

We claim that $||\Lambda_0|| = ||x_0||/d$. For, taking $y_n \in Y$, $||y_n + x_0|| \to d$,

$$\Lambda_0(y_n + x_0) = \|x_0\| \leq \|\Lambda_0\|\|y_n + x_0\| \to \|\Lambda_0\|d,$$

hence $||x0||/d \leq ||\Lambda_0||$.

Now, we apply Hahn-Banach theorem to obtain an extension $\tilde{\Lambda}$ of Λ_0 in X' with

$$||\tilde{\Lambda}|| = ||\Lambda_0|| = ||x0||/d.$$

Then, a constant multiple of $\tilde{\Lambda}$, $d/||x_0||\tilde{\Lambda}$, is our desired functional.

Corollary 5.2.2 . *For any non-zero $x_0 \in X$, there exists $\Lambda \in X'$ such that $\Lambda x_0 = ||x_0||$ and $||\Lambda|| = 1$.*

Proof. Apply Theorem 4.3.4 by taking $Y = \{0\}$.

A bounded linear functional with the properties described in this corollary may be called a "dual point" of $x0$. It may not be unique. For instance, consider $(\mathbb{R}^2, \|\cdot\|_1)$ and the vector $x0 = (1, 0)$. It is readily checked that the two linear functionals $\Lambda_1(x, y) = x$ and $\Lambda_2(x, y) = x + y$ are dual points of (1, 0).

Corollary 5.2.3. For any x ∈ X,

$$\|x\| = \sup_{\Lambda \in X',\ \Lambda \neq 0} \frac{|\Lambda x|}{\|\Lambda\|}.$$

Proof. From $\|\Lambda x\| \leq \|\Lambda\|\|x\|$ we obtain

$$\|x\| \geq \sup_{\Lambda \in X',\ \Lambda \neq 0} \frac{|\Lambda x|}{\|\Lambda\|}.$$

On the other hand, for a given non-zero x, pick Λ^* such that $\|\Lambda^*\| = 1$ and $\Lambda^* x_0 = \|x_0\|$. We have

$$\|x\| = \frac{|\Lambda^* x|}{\|\Lambda^*\|} \leq \sup_{\Lambda \in X',\ \Lambda \neq 0} \frac{|\Lambda x|}{\|\Lambda\|}.$$

This corollary tells us that there are sufficiently many bounded linear functional to determine the norm of any vector. Furthermore, the "sup" in the above expression can be strengthened to "max" as it is attained by Λ^*

5.3 THE DUAL SPACE OF CONTINUOUS FUNCTIONS

The dual space of $C[a, b]$ is described essentially by a representation theorem of Riesz. To formulate it we need to introduce two new concepts: Riemann-Stieltjes integral and functions of bounded variations. Since our focus is on the application of the Hahn-Banach theorem, we simply state basic results (Facts 1 to 4) on these new concepts and leave them as exercises. You may consult Rudin's "Principles of Mathematical Analysis" or [Hewitt-Stromberg] for more on Riemann-Stieltjes integrals.

First of all, for any two complex-valued functions f and g on $[a, b]$ we define its **Riemann-Stieltjes sum** $R(f, g, P)$ with respect to a tagged partition $\dot{P}$ by

$$R(f, g, P) = \sum_{1}^{n} f(z_j)\big(g(x_j) - g(x_{j-1})\big)$$

where $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ is the partition and $z_j \in [x_{j-1}, x_j]$ is a tag. We call f is **Riemann-Stieltjes integrable** with respect to g if there exists $I \in$ F satisfying: For each $\varepsilon > 0$, there exists a δ such that

$$|R(f, g, P) - I| < \varepsilon, \quad \forall P, \quad \|P\| < \delta.$$

Recall that the length of the partition P, $\|P\|$, is given by $\max_{j=1}\{x_j - x_{j-1}\}$. Write $I = \int_a^b f(x)dg(x)$ or simply $\int fdg$ and denote the class of all Riemann-Stieltjes integrable functions by $R_g[a, b]$. Using the definition one can establish the following facts.

Fact 1.

(a) For f_1 and f_2 in $R_g[a, b]$ and $\alpha_1, \alpha_2 \in$ F, we have $\alpha_1 f_1 + \alpha_2 f_2$ belongs to $R_g[a, b]$, and

$$\int (\alpha_1 f_1 + \alpha_2 f_2)dg = \alpha_1 \int f_1 dg + \alpha_2 \int f_2 dg;$$

(b) For $f \in R_{g_1}[a, b] \cap R_{g_2}[a, b]$ and $\alpha_1, \alpha_2 \in$ F, f belongs to $R_{\alpha_1 g_1 + \alpha_2 g_2}[a, b]$ and

$$\int fd(\alpha_1 g_1 + \alpha_2 g_2) = \alpha_1 \int fdg_1 + \alpha_2 \int fdg_2;$$

We will single out a class of g's so that all continuous functions are Riemann-Stieltjes integrable with respect to each of them. A function g

on $[a, b]$ is called a **function of bounded variation** (a BV-function for short) if there exists a constant M such that

$$\sum_{j=1}^{n} \left|g(x_j) - g(x_{j-1})\right| \leq M$$

for all partitions P on $[a, b]$. For a function g of bounded variation, set its **total variation** to be

$$\|g\|_{BV} \equiv \sup\{\sum_{j=1}^{n} \left|g(x_j) - g(x_{j-1})\right| : \text{ all partitions } P\}.$$

It is easy to see that $||g||_{BV}$ satisfies (N2) and (N3) but not (N1), which must be replaced by: "$||g||_{BV} = 0$ implies g is a constant". Nevertheless, we can remove this unpleasant situation by restricting to the subset, $BV_0[a, b]$, consisting of all BV-functions which vanish at a. Then $BV0[a, b]$ forms a normed vector space under $||.||_{BV}$. A by-now routine check shows that it is complete.

Let's look at some examples of BV-functions.

Example. Every monotone function on $[a, b]$ is of bounded variation. In fact, for any P ,

$$\sum_{1}^{n} \left|g(x_j) - g(x_{j-1})\right| = \sum_{1}^{n} \left(g(x_j) - g(x_{j-1})\right) = g(b) - g(a)$$

when g is increasing, so $||g||_{BV} = g(b) - g(a)$. When g is decreasing, $||g||_{BV} = g(a) - g(b)$.

Example. If g is continuously differentiable on $[a, b]$, then $||g||_{BV} \leq ||g'||_{\infty}(b - a)$. For

$$\sum_{1}^{n} \left|g(x_j) - g(x_{j-1})\right| = \sum_{1}^{n} \left|g'(z_j)(x_j - x_{j-1})\right| \leq \|g'\|_{\infty}(b - a)$$

where $z_j \in [x_{j-1}, x_j]$.

Example. Not every continuous function is of bounded variation. Consider the continuous function on $[0, 2/\pi]$ given by

$$h(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Taking the partition P_N to be $\{(n + 1/2)\pi^{-1} : n = 1, \cdots, N\}$ together with the endpoints, one shows that $V h = \infty$ after letting $N \to \infty$.

Fact 2. Every real-valued BV-function can be expressed as the difference of two increasing functions. This is known as Jordan decomposition theorem.

Fact 3. *Every continuous function on* $[a, b]$ *is Riemann-Stieltjes integrable with respect to a BV-function in* $[a, b]$.

In other words, $\int f dg$ is well-defined when $f \in C[a, b]$ and $g \in BV[a, b]$.

Example. Taking $g(x) = x$, the Riemann-Stieltjes integral reduces to the Riemann integral.

Example. Taking g to be continuously differentiable

$$\int f dg = \int f g' dx \text{ and } \|\Lambda_g\| \leq \|g'\|_1.$$

Example. Taking $g = \chi_{[c,b]}$, $a < c < b$, in

$$\sum f(z_j)\big(g(x_j) - g(x_{j-1})\big)$$

all terms vanish except the subinterval $[x_{j-1}, x_j]$ containing c in its interior (we may take a partition in which c is not an endpoint of any subinterval.), so, as $||P|| \to 0$,

$$\int f d\chi_{[c,b]} = f(c).$$

Note that we also have

$$\int f d\chi_{(c,b]} = f(c).$$

Now we come to bounded linear functional on $C[a, b]$. Let's consider two examples. First, fix a point $c \in [a, b]$ and let $\Lambda_1 f = f(c)$. This "evaluation map" is clearly a linear functional with operator norm equal to 1. Next, fix an arbitrary continuous function φ and define

$$\Lambda_2(f) = \int_a^b f(x)\phi(x)dx, \quad f \in C[a, b].$$

$$|\Lambda_2 f| \leq \int_a^b |\phi(x)|dx\|f\|_\infty,$$

Λ_2 is also in the dual of $C[a, b]$. Both functionals can be unified in the setting of Riemann-Stieltjes integrals. Indeed, the first functional corresponds to taking $g = \chi_{[c,b]}$ and the second one to taking g to be a primitive function of φ. In view of this, to every BV-function g we associate it with the functional

$$\Lambda_g f = \int_a^b f\,dg.$$

It is not hard to verify that Λg belongs to the dual space of $C[a, b]$ (see below). Our goal is to show that such association is a norm-preserving linear isomorphism. However, above Example is an obvious obstruction; as both functions $\chi_{[c,b]}$ and $\chi_{(c,b]}$ give the same evaluation map $f(c)$, this association cannot be injective. This difficulty turns out to be minor, and we can overcome it by further restricting the space $BV0[a, b]$.

A function g is called right continuous if $\lim_{h\downarrow 0} g(x + h) = g(x)$. Let

$V[a, b] = \{g \in BV_0[a, b] : \quad g$ is right continuous on $[a, b).\}$

Notice that $\chi_{[c,b]}$ is right continuous but $\chi_{(c,b]}$ is not. It is clear that $V[a, b]$ is a subspace of $BV_0[a, b]$.

Fact 4. (a) *Every BV_0-function g is equal to a unique V-function g* e *except possibly at countably many points.*
(b) $\int f dg = \int f d\widetilde{g}$*, for all* $f \in C[a, b]$.
(c) For $g1, g2 \in V[a, b]$, $\int fdg1 = \int fdg2$ *implies* $g1 = g2$.

Fact 4 (a) can be deduced from a known result in Elementary Analysis, namely, the discontinuity points of a monotone function consist of jump discontinuity and there are at most countable many of them. Since a BV-function is the difference of two increasing functions, the same property

Notes

holds for it. From Facts 1 and 4 we see that the map $g \mapsto \Lambda g$ defines a linear injective map Φ from $V[a, b]$ to $C[a, b]0$. In fact, we have

$$|\Phi(g)f| = |\Lambda_g f| \leq \|g\|_{BV} \ \|f\|_\infty,$$

$$\|\Phi(g)\| \leq \|g\|_{BV}, \quad \forall g \in V[a,b]. \qquad (9)$$

Hence

Here is a version of the Riesz representation theorem.

Theorem 5.3.2. *There is a norm-preserving linear isomorphism from $C[a, b]'$ to $V[a, b]$.*

Proof. The norm-preserving linear isomorphism is, of course, Φ. Let's find its inverse. Let $\Lambda \in C[a, b]0$.

Observing that $C[a, b]$ is a subspace in the normed space $B[a, b]$ of bounded functions, we can use HahnBanach theorem to find an extension Λ e $\in B[a, b]0$ with $||\tilde{\Lambda}|| = ||\Lambda||$. This is crucial!

Our desired inverse g should satisfy $\Lambda f = \Lambda g f$ for $f \in C[a, b]$. Formally,

$$\begin{aligned}\widetilde{\Lambda}(\chi_{[a,c]}) &= \int_a^b \chi_{[a,c]} dg = \int_a^c dg = \int_a^c g'(x)dx \\ &= g(c) - g(a) = g(c),\end{aligned}$$

as g vanishes at a. Motivated by this, we define

$$g(x) = \widetilde{\Lambda}(\chi_{[a,x]}), \quad x \in (a,b],$$

and $g(0) = 0$. We claim that $g \in BV_0[a, b]$ and $||g||_{BV} \leq ||\Lambda||$.

For, with respect to an arbitrary partition P ,

$$\begin{aligned}
\sum_1^n |g(x_j) - g(x_{j-1})| &= \sum_1^n e^{i\theta_j}(g(x_j) - g(x_{j-1})) \\
&\quad \text{(for any } z \in \mathbb{C}, \text{ there exists } e^{i\theta} \text{ such that } |z| = e^{i\theta} z) \\
&= e^{i\theta_1} g(x_1) + \sum_2^n e^{i\theta_j}(g(x_j) - g(x_{j-1})) \\
&= e^{i\theta_1} \tilde{\Lambda}\chi_{[a,x_1]} + \sum_2^n e^{i\theta_j}(\tilde{\Lambda}\chi_{[a,x_j]} - \tilde{\Lambda}\chi_{[a,x_{j-1}]}) \\
&= \tilde{\Lambda}(e^{i\theta_1}\chi_{[a,x_1]}) + \sum_2^n \tilde{\Lambda}(e^{i\theta_j}\chi_{(x_{j-1},x_j]}) \\
&= \tilde{\Lambda}(e^{i\theta_1}\chi_{[a,x_1]} + \sum_2^n e^{i\theta_j}\chi_{(x_{j-1},x_j]}).
\end{aligned}$$

Let's denote the function inside the above bracket by h. Noting that for all $x \in [a, b]$, there exists a unique subinterval $[a, x1]$ or $(xj-1, xj]$ containing x, so $h(x) = e^{i\theta_1}$ or $e^{i\theta_{j0}}$ for some $j0$. In any case $|h(x)| = 1$. It follows that $||h||_\infty = 1$ and that

$$\sum_1^n |g(x_j) - g(x_{j-1})| \le \|\tilde{\Lambda}\| \|h\|_\infty = \|\Lambda\|$$

for the partition P , so $g \in BV_0[a, b]$ and $||g||_{BV} \le ||\Lambda||$.

We define $\Psi : C[a, b]0 \to V [a, b]$ by $\Psi(\Lambda) = \tilde{g}$ where $\tilde{g}$ is the right continuous modification of g

satisfying $\tilde{g}(a) = 0$ defined above. The estimate $||\tilde{g}||_{BV} = ||g||_{BV} \le ||\Lambda||$ can be written as

$$\|\Psi(\Lambda)\|_{BV} \le \|\Lambda\|, \quad \forall \Lambda \in C[a, b]'. \qquad (10)$$

To complete the proof, we claim that $\Lambda f = \Lambda \tilde{g} f$, for all $f \in C[a, b]$. It means $\Phi(\Psi(\Lambda)) = \Lambda$ on $C[a, b]0$. In particular, Φ is surjective. Moreover, from (10) and (9) we have

$$\|g\|_{BV} \le \|\Phi(g)\| \le \|g\|_{BV},$$

so Φ is norm-preserving.

It remains to verify $\Lambda f = \Lambda g f$. Given $\varepsilon > 0$, since f is Riemann-Stieltjes integrable with respect to

g, there is some $\delta 1$ such that

$$\left| \int_a^b f dg - \sum_1^n f(x_j)(g(x_j) - g(x_{j-1})) \right| < \varepsilon,$$

for P , $kP\ k < \delta 1$. Using

$$\begin{aligned}\sum_1^n f(x_j)(g(x_j)-g(x_{j-1})) &= f(x_1)g(x_1)+\sum_2^n f(x_j)(g(x_j)-g(x_{j-1}))\\ &= f(x_1)\tilde{\Lambda}\chi_{[a,x_1]}+\sum_2^n f(x_j)\tilde{\Lambda}\chi_{(x_{j-1},x_j]}\\ &= \tilde{\Lambda}\big(f(x_1)\chi_{[a,x_1]}+\sum_2^n f(x_j)\chi_{(x_{j-1},x_j]}\big)\\ &\equiv \tilde{\Lambda}(f'),\end{aligned}$$

$$f'(x)=f(x_1)\chi_{[a,x_1]}+\sum_2^n f(x_j)\chi_{(x_{j-1},x_j]}(x),$$

$$\left|\int_a^b f dg-\tilde{\Lambda}(f')\right|<\varepsilon.$$

(11) As f is uniformly continuous on $[a, b]$, for every $\varepsilon > 0$, there is some $\delta 2$ such that

$$|f(x) - f(y)| < \varepsilon, \qquad \text{for all } x, y, |x - y| < \delta 2.$$

We take $\delta = \min\{\delta 1, \delta 2\}$ and $kP\ k < \delta$. Then

$$\begin{aligned}f(x)-f'(x) &= f(x)\big[\chi_{[a,x_1]}(x)+\sum_2^n \chi_{(x_{j-1},x_j]}(x)\big]-f'(x)\\ &= (f(x)-f(x_1))\chi_{[a,x_1]}(x)+\sum_2^n (f(x)-f(x_j))\chi_{(x_{j-1},x_j]}(x).\end{aligned}$$

As each x belongs to exactly one subinterval, say, the $j0$-th,

$$|f(x) - f'(x)| = |f(x) - f(x_{j0})| < \varepsilon$$

if $||\ P\ || < \delta$. That means, $||f - f'||\infty < \varepsilon$, so

$$|\tilde{\Lambda} f - \tilde{\Lambda} f'| \leq \|\tilde{\Lambda}\| \|f - f'\|_\infty < \varepsilon \|\tilde{\Lambda}\|.$$

Combining with (11),

$$\left| \int_a^b f dg - \tilde{\Lambda} f \right| \leq \left| \int_a^b f dg - \tilde{\Lambda} f' \right| + \left| \tilde{\Lambda} f' - \tilde{\Lambda} f \right| \leq (1 + \|\Lambda\|)\varepsilon.$$

Since ε is arbitrary and Λ extends $\tilde{\Lambda}$, $\int fdg = \tilde{\Lambda} f = \Lambda f$.

The proof of Theorem is completed.

CHECK YOUR PROGRESS

1. Prove - *For any non-zero* $x_0 \in X$*, there exists* $\Lambda \in X'$ *such that* $\Lambda x_0 = ||x_0||$ *and* $||\Lambda|| = 1$.

2. Define **Riemann-Stieltjes sum & Riemann-Stieltjes integrable.**

5.4 REFLEXIVE SPACES

To any normed space X there associates another normed space, namely its dual X'. Since the dual space X' is again a normed space, one may consider the double dual space $(X')'$ or simply X''. It is interesting to observe that any vector in X can be viewed as a vector in X''.

Proposition 5.4.1. *For* $x_0 \in X$*, define a functional* $\widetilde{x_0}$ *on* X' *by*

$$\widetilde{x_0}\,(\Lambda) = \Lambda x_0, \qquad \forall \Lambda \in X'.$$

Then $\widetilde{x_0} \in X''$ and $\|\widetilde{x_0}\| = \|x_0\|$. The mapping J (called canonical identification or canonical

embedding): $x_0 \mapsto \widetilde{x_0}$ is a norm-preserving, linear map from X to X".

Notice here $||x_0||$ is the norm of x_0 in X and $||\widetilde{x_0}||$ stands for the operator norm in X''.

Proof. Clearly, $J : x_0 \mapsto \widetilde{x_0}$ is linear. From

$$|\tilde{x_0}(\Lambda)| = |\Lambda x_0| \leq \|\Lambda\| \|x_0\|$$

we also have $\widetilde{x_0} \in X''$ with operator norm $||\widetilde{x_0}|| \leq ||x_0||$. By Corollary 3.10 we pick Λ0 satisfying $||\Lambda_0|| = 1$ and $\Lambda_0 x_0 = ||x_0||$. Then

$$\|x_0\| = \Lambda_0 x_0 = \tilde{x_0}(\Lambda_0) \leq \|\tilde{x_0}\| \|\Lambda_0\| = \|\tilde{x_0}\|,$$

so J is norm-preserving.

A normed space is called a **reflexive space** if the canonical identification is a norm-preserving linear isomorphism. By Proposition 3.13 subjectivity of the canonical map is sufficient for the space to be reflexive. Interestingly there are non-flexible Banach spaces with the property that there exists a norm preserving linear isomorphism from X to X''. Of course, this isomorphism cannot be the canonical identification. It is an easy exercise to show that all finite dimensional normed spaces are reflexive.

Proposition 5.4.2. ℓ^p $(1 < p < \infty)$ *is a reflexive space.*

Proof. For every $T \in (\ell^p)0$, there exists a unique $y^T \in \ell^q$ such that $T\,x = \sum_j y_j^T x_j$ for all x in ℓ^p.

Given $\Lambda \in (\ell^p)''$, the linear functional given by $\Lambda_1 y^T = \Lambda T$ is bounded in ℓ^q.

There exists some $z \in \ell^p$ such that $\Lambda T = \Lambda_1 y^T = \sum_j y_j^T z_j$ for all yT in ℓ^q.

Recalling from the definition the canonical identification of z, $z^*(T) = T\,z = \sum_j y_j^T z_j$.

By comparison we see that $\Lambda = z*$, that is to say, the canonical identification is surjective, so `p is reflexive.

Likewise, the L^p-space $L^p(X, \mu)$ where (X, μ) is a measure space and $p \in (1, \infty)$ is reflexive. This result, also known as Riesz representation theorem, is a standard one in real analysis, see, for instance, [Hewitt-Stromberg] or [Rudin].

Before giving some non-reflexive spaces, we note two results which may be viewed as necessary conditions for reflexivity.

Proposition 5.4.3. *A reflexive space is a Banach space.*

Proof. We know that, the dual space of a normed space is a Banach space. As now $X = (X')'$ is the dual of the normed space X', it must be complete.

From this result, we see that $(C[a, b], ||\cdot||p)$ is not reflexive for $p \in [1, \infty)$ since $C[a, b]$ is not complete under the L^p-norm.

Proposition 5.4.4 . *If X' is separable, then X is also separable.*

Proof. As X' is separable, the subset $\{\Lambda \in X' : ||\Lambda|| = 1\}$ is also separable. Pick a countable dense set $\{\Lambda k\}$ in this subset. Using the definition of the operator norm, for each Λk we can find xk, $||x_k|| = 1$, such that $\Lambda_k x_k \geq 1/2$. Let E be the closure of the span of $\{x\}_1^\infty$. E is separable because all linear combinations of xk's

with coefficients in Q or Q + iQ form a countable dense subset in E. We shall finish the proof by showing $E = X$.

For, if $X \setminus E \neq \emptyset$ we pick $x0 \in X \setminus E$. We can find some $\Lambda_0 \in X'$ such that $\Lambda_0 = 0$ on E, $||\Lambda_0|| = 1$. On the other hand, as $\{\Lambda k\}$ is dense, for any $\varepsilon < 1/2$, there is some k_0 such that $||\Lambda_0 - \Lambda_{k0}|| < \varepsilon$. It follows that for all $x \in E$, $||x|| = 1$.

$$\begin{aligned} |\Lambda_{k_0} x| &\leq |(\Lambda_{k_0} - \Lambda_0)x| + |\Lambda_0 x| \\ &= |(\Lambda_{k_0} - \Lambda_0)x| \\ &\leq \|\Lambda_{k_0} - \Lambda_0\| < \varepsilon. \end{aligned}$$

Taking $x = x_{k0}$,

$$\frac{1}{2} \leq |\Lambda_{k_0} x_{k_0}| < \varepsilon,$$

contradiction holds.

Using Proposition 3.16, we see that `1 is not reflexive. For, if it is, then $(\ell^{\infty})' = (\ell^1)00 = \ell^1$. As ℓ^1
is separable, ℓ^{∞} must be separable. Similarly it is not hard to show that the dual of $C[a, b]$ is not separable, so $C[a, b]$ is not reflexive.

Reflexive spaces have many nice properties. They arise from many contexts, for instance, the Sobolev spaces $W^{k,p}(\mathbb{R}^n)$, $1 < p < \infty$, are an indispensable tool in the modern study of partial differential equations. They reflexive and separable. We point out three further properties of a reflexive space:

- ✓ First, any closed subspace of a reflexive space is also a reflexive space.

- ✓ Second, a Banach space is reflexive if and only if its dual is reflexive. The proofs of these two results are elementary and left as exercises.

- ✓ Third, the best approximation problem always has an affirmative answer in a reflexive space. More precisely, let C be a closed, convex subset in this space and x_0 a point lying outside C. Then there exists a point z_0 in C such that $||x - z_0|| \leq ||x - z||$for all $z \in C$.

CHECK YOUR PROGRESS

3. Establish two facts from the definition of Dual Space of Continuous function.

4. State the properties of Reflexive Spaces

5.5 LET'S SUM UP

All these theorems relate the space of continuous linear functionals to integrals with respect to the space of certain measures in different contexts. Riesz representation theorem is significant because it links up real analysis and functional analysis.

5.6 KEYWORDS

Isomorphism - The word derives from the Greek iso, meaning "equal," and morphosis, meaning "to form" or "to shape." Formally, an **isomorphism** is bijective morphism. Informally, an **isomorphism** is a map that preserves sets and relations among elements.

Primitive Function - In calculus, an antiderivative, **primitive function**, **primitive** integral or indefinite integral of a **function** f is a differentiable **function** F whose derivative is equal to the original **function** f.

Subinterval - an interval that is a subset of a given interval.

5.7 QUESTION FOR REVIEW

1. Prove Fact 1 in Section 5.2

2. Show that $C[a, b] \subset R_g[a, b]$ for any monotone function g on $[a, b]$.

3. Let g be an increasing function and $\tilde{g}$ the right continuous function obtained from g. Prove that

$$\int f \, d\tilde{g} = \int f \, dg, \quad \forall f \in C[a, b].$$

4. Prove that any BV -function g can be written as the difference of two increasing functions. This is Jordan decomposition theorem. At each $x \in [a, b]$, define the increasing function Ng by

$$N_g(x) = \sup \sum_j |g(x_{j+1}) - g(x_j)|$$

where the supremum is over all partitions of $[a, x]$. Show that $Ng(y) \geq Ng(x) + |g(x) - g(y)|$, for $y > x$.

5.8 SUGGESTED READINGS

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5.9 ANSWER TO CHECK YOUR PROGRESS

1. Provide proof – 5.1.2

2. Provide definition and representation – 5.2

3. Provide statements of two facts with briefs – 5.2

4. Provide statement given in the last section of 5.3.4.

UNIT 6: BOUNDED LINEAR OPERATOR I

STRUCTURE

6.0 OBJECTIVE

Understand the concept of Bounded Linear Operators

Enumerate Examples of Linear Operators

Comprehend Baire Theorem.

6.1 INTRODUCTION

We studied normed spaces in the previous three chapters. Now we come to bounded linear operators on these spaces. A bounded linear operator is the infinite dimensional analog of a matrix. The norm preserving linear isomorphism and the canonical identification studied the previous chapters are special cases of bounded linear operators. They are very special ones. Due to the complexity of the structure of infinite dimensional spaces, bounded linear operators are much more diverse and difficult to investigate than matrices, and yet there are many applications. After introducing basic definitions and properties in Section 1 and examining some examples in Section 2, we turn to two theorems, namely, the uniform boundedness principle and the open mapping theorem. Together with Hahn-Banach theorem, they form the cornerstone of the subject. Nevertheless, unlike the Hahn-Banach theorem, both theorems depend critically on completeness. Being the infinite dimensional counterpart of the eigenvalues of a matrix, spectra play an important role in analyzing bounded linear operators.

6.2 BOUNDED LINEAR OPERATORS

Let X and Y be two vector spaces over F. Recall that a map $T : X \to Y$ is a linear operator (usually called a linear transformation in linear algebra) if for all $x1$, $x2 \in X$ and $\alpha, \beta \in$ F,

$$T(\alpha x1 + \beta x2) = \alpha T(x1) + \beta T(x2).$$

The null space (or kernel) of T, $N(T)$, is the set $\{x \in X : T x = 0\}$ and the range of T is denoted by $R(T)$. Both $N(T)$ and $R(T)$ are subspaces of X and Y respectively.

The collection of all linear operators from X to Y forms a vector space $L(X, Y)$ under point wise addition and scalar multiplication of functions.

When $X = \mathbb{F}^n$ and $Y = \mathbb{F}^m$, any linear operator (or called linear transformation) can be represented by an $m \times n$ matrix with entries in F. The vector space $L(\mathbb{F}^n, \mathbb{F}^m)$ is of dimension mn. When X and Y are normed, one prefers to study continuous linear operators. $T \in L(X, Y)$ is continuous means it is continuous as a mapping from the metric space X to the metric space Y. It is called a bounded linear operator if it maps any bounded set in X to a bounded set in Y. By linearity, it suffices to map a ball to a bounded set.

Proposition 6.2.1. *Let $T \in L(X, Y)$ where X and Y are normed spaces. We have*

(a) T is continuous if and only if it is continuous at a point.

(b) T is bounded if and only if there exists a constant $C > 0$ *such that*

$$||Tx|| \leq C||x||, \qquad \text{for all } x.$$

(c) T is continuous if and only if T is bounded.

We denote the collection of all bounded linear operators from X to Y by $B(X, Y)$. It is a subspace of $L(X, Y)$. They coincide when X and Y are of finite dimension, of course. We observe that $X' = B(X, F)$. The space $B(X, Y)$ not only inherits a vector space structure from X and Y but also a norm structure.

For $T \in B(X, Y)$, define its **operator norm** by

$$\|T\| \equiv \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\|.$$

It is immediate to check that $k{\cdot}k$ makes $B(X, Y)$ into a normed space. Furthermore, for $T \in B(X, Y)$ and $S \in B(Y, Z)$, the composite operator $ST \in B(X, Z)$ and

$$||ST|| \leq ||S||\,||T||.$$

Taking $X = Y = Z$, it means we have a multiplication structure on $B(X)$ which makes $B(X)$ a Banach algebra. Banach algebra is an advanced topic which has many applications in abstract harmonic analysis.
The following proposition is useful in determining the operator norm.

Proposition 6.2.2. *Let* $T \in B(X, Y)$*. Suppose M is a positive number satisfying*

(a) $||Tx|| \leq M||x||$, *for all* $x \in D$ *where D is a dense set in X, and*

(b) there exists a nonzero sequence $\{xk\} \subset D$ *such that* $||T_{x_k}||/||x_k|| \to M$

.

Then $M = ||T||$.

Proof. For any $x \in X$, pick a sequence $yk \to x$, $yk \in D$. Then

$$||Tx|| = \lim_{k\to\infty} ||T_{y_k}|| \leq M \lim_{k\to\infty} ||y_k|| = M||x||$$

shows that

$$||Tx|| \leq M||x||, \qquad \text{for all } x \in X.$$

By the definition of the operator norm,

$$\|T\| \leq \sup_{\|x\|=1} \|Tx\| \leq M.$$

On the other hand, for the sequence *{xk}* given in (b),

$$M = \lim_{k\to\infty} \frac{\|Tx_k\|}{\|x_k\|} \leq \|T\|,$$

so $M = ||T||$.
The following result, which generalizes Proposition 3.4, can be established in a similar way.

Proposition 6.2.3. *$B(X, Y)$ is a Banach space if Y is a Banach space.*
Let $T \in B(X, Y)$ where X and Y are normed spaces. Then T is called **invertible** if it is bijective with the inverse in $B(Y, X)$. When X and Y are finite dimensional, every linear bijective map is automatically bounded, so it is always invertible. However, this is no longer true in the infinite dimensional setting.
In many applications, some problem can be rephrased to solving the equation $Tx = y$ in some spaces for some linear operator T . The invertibility of T means the problem has a unique solution for every y. Furthermore, for two solutions $T_{x_i} = y_i$, $i = 1, 2$, the continuity of T^{-1} implies the estimate $||x_2 - x_1|| \leq C||y_2 - y_1||$, $C = ||T -1||$, from which we see that the solution depends continuously on the given data.
This is related to the concept of well-posedness in partial differential equations.
The following general result is interesting.

Theorem 6.2.4. *Let $T \in B(X, Y)$ be invertible where X is a Banach*

Notes

space. Then $S \in B(X, Y)$ *is invertible whenever* S *satisfies* $||I - T^{-1}S||$, $||I - ST^{-1}|| < 1$.

The conditions $||I - T^{-1}S||$, and $||I - ST^{-1}|| < 1$ should be understood as a measurement on how

S is close to T . The idea behind this theorem as follows. We would like to solve $Sx = y$ for a given y. Rewriting the equation in the form $Tx + (S - T)x = y$ and applying the inverse operator to get $(I - E)x = T^{-1} y$ where I is the identity operator on $B(X, X)$ and $E = T^{-1}(T - S) \in B(X, X)$ is small in operator norm. So the solution x should be given by $(\sum_{j=0}^{\infty} E^j)$ $T^{-1} y$ as suggested by the formula $(1 - x)^{-1} = \sum_j x^j)$ for $|x| < 1$.

Our proof involves infinite series in $B(X, X)$. As parallel to what is done in elementary analysis, an infinite series $\sum_k x^k \in (X, ||\cdot||)$, is **convergent** if its partial sums $s_n = \sum_1^n x_k$ form a convergent sequence in $(X, ||\cdot||)$. We note the following criterion, “M-Test”, for convergence.

Proposition 6.2.5. An infinite series $\sum_k x^k$ in the Banach space X is convergent if there exist $a_k \geq 0$ such that $||xk|| \leq a_k$ for all k and $\sum_k a^k$ is convergent.

Proof. We have

$$\|s_n - s_m\| = \| \sum_{m+1}^{n} x_k \| \leq \sum_{m+1}^{n} \|x_k\| \leq \sum_{m+1}^{n} a_k,$$

and the result follows from the convergence of $\sum_k a^k$ and the completeness of X.

In particular, the series is convergent if there exists some $\rho \in (0, 1)$ such that $kxkk \leq \rho k$ for all k.

Corollary 6.2.6. *Let* $L \in B(X, X)$ *where* X *is a Banach space with* $||L|| <$ 1. *Then* $I - L$ *is invertible with inverse given by*

$$(I - L)^{-1} = \sum_{k=0}^{\infty} L^k.$$

Proof. By assumption, there exists some $\rho \in (0, 1)$ such that $||L|| \leq \rho$. From $||L^k|| \leq ||L||^k \leq \rho k$ and Proposition 6.1.5 that $\sum_{k=0}^{\infty} L^k$ converges in $B(X, X)$. Moreover,

$$(I-L)\sum_{k=0}^{\infty}L^k=\sum_{k=0}^{\infty}(I-L)L^k=\lim_{n\to\infty}\sum_{k=0}^{n}(I-L)L^k=\lim_{n\to\infty}(I-L^{n+1})=I.$$

Similarly, $\sum_{k=0}^{\infty} L^k (I-L) = I$.

Proof of Theorem 6.2.4. We adopt the notations in the above paragraph. As $||E|| < 1$ by assumption, Corollary 6.1.6 implies that $\sum_{j=0}^{\infty} E^j$ is the inverse of $I - E$. Letting $x = (\sum_{j=0}^{\infty} E^j)T^{-1} y$, then $(I - E)x = T^{-1} y$, that is, $Sx = y$. We have shown that S is onto. Also it is bounded. On the other hand, from

$$\|(S-T)x\|=\|(ST^{-1}-I)Tx\|\le\|ST^{-1}-I\|\|Tx\|,$$

we have

$$\begin{aligned}\|Sx\| &\geq \big|\|Tx\|-\|(S-T)x\|\big|\\ &\geq (1-\|I-ST^{-1}\|)\|Tx\|\\ &\geq \frac{(1-\|I-ST^{-1}\|)}{\|T^{-1}\|}\|x\|,\end{aligned}$$

So S has a bounded inverse. We have completed the proof of this theorem.

As an application let us show that all invertible linear operators form an open set in $B(X, Y)$ when X is complete. Let T_0 be invertible. Then for each T satisfying $||T - T_0|| < \rho \equiv 1/||T_0 -1||$, we have

$$||I - T_0^{-1}T|| \leq ||T_0^{-1}1||||T_0 - T|| < 1,$$

so by this theorem T is invertible. That means the ball $B\rho\ (T0)$ is contained in the set of all invertible linear operators, and consequently it is open. For an $n \times n$-matrix, its corresponding linear transformation is invertible if and only if it is nonsingular. Again a matrix is nonsingular if and only if its determinant is non-zero. As the determinant is a continuous function on matrices (as the space Fn2), for all matrices close to a nonsingular matrix their determinants are non-zero, so all nonsingular matrices form an open set in the vector space of all $n \times n$-matrices. Theorem 6.1.4 shows that this result holds in general.

Notes

A main theme in linear algebra is to solve the nonhomogeneous linear system

$$Ax = b,$$

where A is an $m \times n$ matrix and $b \in \mathbb{R}m$ are given. The Fredholm alternative states that either this linear system is uniquely solvable, or the homogeneous system

$$A'y = 0,$$

has nonzero solutions y, where $A0$ is the transpose matrix of A. Moreover, when this happens, the non-homogeneous system is solvable if and only if b is perpendicular to all solutions y of the homogeneous system. Can we extend this beautiful result to linear operators in Banach spaces? We need to answer the following question before we can proceed, namely, how do we define the transpose of a linear operator?

For a bounded linear operator T from the normed space X to another normed space Y there associates with a linear operator T' from Y' to X' called the **transpose** of T. Indeed, we define T' by

$$T'y'(x) \equiv y'(Tx), \quad \text{for all } y' \in Y', \quad x \in X.$$

It is straightforward to prove the following result.

Proposition 6.2.7. *Let T' be defined as above. Then*

(a) T' is a bounded linear operator from Y' to X'. Furthermore, $||T'|| = ||T||$.

(b) The correspondence $T \to T'$ is linear from $B(X, Y)$ to $B(Y', X')$.

(c) If $S \in B(Y, Z)$ where Z is a normed space, then $(ST)' = T'S'$.

We examine the finite dimensional situation. Let T be a linear operator from $\mathbb{F}^n$ to $\mathbb{F}^m$. Let $\{ej\}$ and $\{fj\}$ be the canonical bases of $\mathbb{F}^n$ and $\mathbb{F}^m$ respectively. We have $Tx = \sum a_{k_j} \alpha_j f_k$ where $x = \sum_j \alpha_j e_j$, so T is represented by the matrix $m \times n$-matrix (a_{k_j}). On the other hand, we represent T' as a matrix with respect to the dual canonical bases $\{f_j'\}$ and $\{e_j'\}$ } as $T'y' = \sum b_{k_j} \beta_j e_k'$ where $y' = \sum_j \beta_j f_j'$.

From the relation $T'y'(ej) = y'(T_{e_j})$ for all j we have $b_{k_j} = a_{j_k}$. Thus the matrix of T' is the transpose of the matrix of T. This justifies the terminology. In some books it is called the adjoint of T. Here we shall reserve this terminology for a later occasion.

There are close relations between the ranges and kernels of T and those of its transpose which now we explore. Recall that the kernel of $T \in B(X, Y)$ is given by $N(T) = \{x \in X : Tx = 0.\}$ and its rangeis $R(T) \equiv T(X)$. The null space is always a closed subspace of X and $R(T)$ is a subspace of Y, but it may not be closed.

For a subspace Y of the normed space X, we define its **annihilator** to be

$$Y^{\perp} = \{x' \in X' : x'(y) = 0, \quad \text{for all } y \in Y\}.$$

Similarly, for a subspace G of X', its **annihilator** is given by

$$^{\perp}G = \{x \in X : x'(x) = 0, \quad \text{for all } x' \in G\}.$$

It is clear that the annihilators in both cases are closed subspaces, and the following inclusions hold:

$$Y \subset^{\perp} (Y^{\perp}),$$

$$G \subset (^{\perp}G)^{\perp}.$$

Lemma 6.2.8. *Let X be a normed space, Y a closed subspace of X and G a closed subspace of X0. Then*

(a) $Y =^{\perp} (Y^{\perp})$;

(b) in addition, if X is reflexive,

$$G = (^{\perp}G)\perp.$$

Proof. (a) It suffices to show $^{\perp}(Y^{\perp}) \subset Y$. Any $x0 \in {}^{\perp}(Y^{\perp})$ satisfies $\Lambda x_0 = 0$ whenever Λ vanishes on Y. x_0 belongs to Y.

(b) It suffices to show $^{\perp}(G^{\perp}) \subset G$. Any $\Lambda_1 \in (\perp G)\perp$ satisfies $\Lambda_1 x = 0$ for all $x \in \perp G$. If $\Lambda 1$ does not belong to G, as G is closed and the space is reflexive, there is some $x1 \in X$ such that $\Lambda_1 x_1 \neq 0$ and $x_1 \in {}^{\perp}G$ contradiction holds.

Notes

Proposition 6.1.9. *Let X and Y be two normed spaces and $T \in B(X, Y)$. Then we have*

$$N(T') = \overline{R(T)}^{\perp},$$

$$N(T) = {}^{\perp}\overline{R(T')},$$

$$^{\perp}N(T') = \overline{R(T)},$$

$$N(T)^{\perp} = ({}^{\perp}\overline{R(T')})^{\perp}.$$

***Proof*.** $T'y_0' = 0$ means $T'y_0'(x) = 0$ for all $x \in X$. By the definition of the transpose of T we have $y_0'(Tx) = 0$ for all x. Since T is continuous, $y_0' \in \overline{R(T)}^{\perp}$. We conclude that $N(T') \subset \overline{R(T)}^{\perp}$. By reversing this reasoning we obtain the other inclusion, so the first identity holds. The second identity can be proved in a similar manner. The third and the fourth identities are derived from the first and the second after using the previous lemma. It is clear that we have

Corollary 6.2.10. *Let X and Y be normed and $T \in B(X, Y)$. Then $R(T)$ is dense in Y if and only if T'is injective.* The significance of this result is evident. It shows that in order to prove the solvability of the equation $Tx = y$ for any given $y \in Y$, it suffices to show that the only solution to $T'y' = 0$ is $y' = 0$. This sets up a relation between the solvability of the equation $Tx = y$ and the uniqueness of the transposed equation $T'x = 0$. Fredholm alternative can be established for linear operators with more structure. For instance, in Chapter 6 we will show that it holds for $T = Id + K$ where K is a compact operator on a Hilbert space.

6.3 EXAMPLES OF LINEAR OPERATORS

There are plenty linear operators in analysis. Here we discuss some examples.

Linear operators on sequence spaces are direct generalization of linear transformations on $\mathbb{F}^n$. Let

$x = (x1, x2, \cdots)$ be a sequence. Then $Tx = (y1, y2, \cdots)$ is again a sequence whose entries y_k depends linearly on x. You may write it formally as

$$y_k = \sum_{j=1}^{\infty} c_{jk} x_j.$$

Depending on which sequence space and the growth on the coefficients c_{jk}, T defines a bounded linear operator or an unbounded one.

Let us consider two cases. First, let $\{aj\}$ be a null sequence with nonzero terms and define a linear operator from ℓ^p to itself by $Tx \equiv (a1x1, a2x2, \cdots,)$. It is clear that $|Tx|p \leq ||a||_\infty ||x||_p$ so T is bounded.

However, it is not invertible because its inverse is not bounded. To see this, assume $T-1$ exists. But then $T^{-1} ej = a_j^{-1} e_j$ which implies that $|a_j^{-1}|$'s are uniformly bounded, contradicting that $\{aj\}$ is null.

Second, the shift (to the right) operator $S_R : \ell^p \mapsto \ell^p$ $(1 \leq p \leq \infty)$ given by

$$S_R(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots).$$

It is easily checked that $SR \in B(\ell^p, \ell^p)$ and $kSRk = 1$. Obviously SR is not onto, so it is not invertible.

$$\mathscr{I} f(x) = \int_a^b K(x,y) f(y) dy, \quad \text{for all } f \in C[a,b].$$

Now, we consider integral operators. These operators arise as the inverse operators for differential operator as well as convolution operators. We restrict our attention on the one dimensional situation.

Fix a continuous function $K \in C([a, b] \times [a, b])$ (usually called the integral kernel) and define

Clearly I is linear. Let's show that it is also bounded on $C[a, b]$. In fact,

$$|\mathscr{I} f(x)| \leq \int_a^b |K(x,y)||f(y)| dy \leq M\|f\|_\infty$$

where $M = \sup_x \int_a^b |K(x,y)|\, dy$. So $||I|| \leq M$. By some careful work, one can show that $||I||$ is equal to M precisely.

Notes

Integral operators can also be defined in other spaces. To do this we note the following lemma.

Lemma 6.3.1. *L*et X and Y be Banach spaces and T is a linear operator from X1 to Y where X1 is a dense subspace of X. Suppose that there is a constant C such that

$$||Tx|| \leq C||x||, \text{ for all } x \in X1.$$

Then T can be uniquely extended to a bounded linear operator from X to Y whereas the above estimate holds on X.

We shall make no difference between *T* and its extension. We apply this lemma to the *Lp*-spaces. From the estimate

$$\begin{aligned} \int_a^b \Big| \int_a^b K(x,y)f(y)dy \Big|^p dx &\leq (b-a) \max |K|^p \Big(\int_a^b |f(y)|dy \Big)^p \\ &\leq (b-a)^p \max |K|^p \Big(\int_a^b |f(y)|^p dy \Big), \end{aligned}$$

we see that *I* can be extended to become a bounded linear operator on *Lp*[*a, b*] by the lemma. Although the integral

$$\int_a^b K(x,y)f(y)dy$$

may not make sense for the "ideal points" in $L^p[a, b]$, it is customary to denote it by the same expression for all points in this space.

In passing one should note that the abuse of notation *I* ; *I f* first stands for $f \in C[a, b]$ but then for its extension in $L^p(a, b)$ for all *p*.

What is the transpose of *I* ? Let us determine it on *L*2(*a, b*). From real analysis we know that this space is self-dual, that is, any bounded linear functional on *L*2(*a, b*) is given by

$$\Lambda_g(f) = \int_a^b f(x)g(x)dx,$$

for some $g \in L^2(a, b)$, that is, the map $\Phi : L^2(a, b) \to L^2(a, b)'$ given by $g \mapsto \Lambda g$ is a norm-preserving linear isomorphism. Now, from the definition for the transpose,

$$(\mathscr{I}'\Lambda_g)(f) = \Lambda_g(\mathscr{I}f) = \int_a^b \Big(\int_a^b K(x,y)f(y)dy \Big) g(x)dx = \int_a^b h(x)f(x)dy,$$

where

$$h(x) = \int_a^b K(y,x)g(y)dy.$$

Hence $I'\Lambda_g = \Lambda_h$. Since $L2(a, b)0$ may be identified with $L^2(a, b)$ via Φ, the transpose of I may be viewed as a map on $L2(a, b)$ to itself given by

$$\mathscr{I}'f(x) = \int_a^b K(y,x)f(y)dy.$$

In particular, $I = I'$ when $K(x, y)$ is symmetric in x and y.

Now given a continuous function f in $C(S1)$ (the space of all continuous, $2\pi-$periodic functions) we can define a sequence of complex numbers by its Fourier coefficients

$$c_n = \int_0^{2\pi} f(x)e^{-inx}dx, \quad \text{for } n \in \mathbb{Z}.$$

The Parseval identity

$$\sum_{n\in\mathbb{Z}} |c_n|^2 = \int_0^{2\pi} |f(x)|^2 dx$$

shows that the linear operator $F : (C(S^1), ||\cdot||_2) \to \ell^2(\mathbb{Z})$ assigning f to $\{c_n\}$ can be extended to become a norm preserving linear isomorphism from $L^2(S^1)$ to ℓ^2. In particular, F is invertible. This result partly justifies the assertion that a function is determined by its Fourier series. In the study of the well-posedness of solutions of partial differential equations we encounter numerous linear operators. This provides opportunity to apply the soft method of functional analysis to partial differential equations. Very often it is crucial to find the most appropriate spaces for the differential operator or its inverse operator to act on.

Notes

Finally, let's consider the differential operator. Let X be the subspace of $C[0, 1]$ consisting of continuous differentiable functions. The differential operator d/dx maps X to $C[0, 1]$. It is linear but unbounded.

Taking $f_k = \sin kx$, $||df_k/dx||_\infty = ||k \cos kx||_\infty = k \to \infty$, but $||f_k||_\infty = 1$..

CHECK YOUR PROGRESS

1. Define Bounded Linear Operator

2. What is **annihilator?**

6.4 BAIRE THEOREM

In the next two sections we shall discuss the uniform boundedness principle and the open mapping theorem both due to Banach. The underlying idea of the proofs of these theorems is the Baire theorem for complete metric spaces.

The motivation is somehow a bit strange at first glance. It is concerned with the decomposition

of a space as a union of subsets. For instance, we can decompose the plane $\mathbb{R}^2$ as the union of strips $\mathbb{R}^2 = \bigcup_{k\in Z} S_k$ where $S_k = (k, k + 1] \times \mathbb{R}$. In this decomposition each S_k is not so sharply different from $\mathbb{R}^2$ Aside from the boundary, the interior of each Sk is just like the interior of R2. On the other hand, one can make the more extreme decomposition: $\mathbb{R}^2 = \bigcup_{\alpha\in\mathbb{R}} \ell_\alpha$ where $l\alpha = \{\alpha\} \times \mathbb{R}$. Each $l\alpha$ is a vertical straight line and is very different from $\mathbb{R}^2$. It is simpler in the sense that it is one-dimensional and has no area. The sacrifice is now we need an uncountable union. The question is: Can we represent $\mathbb{R}^2$ as a countable union of these sets (or sets with lower dimension)? It turns out that the answer is no. The obstruction comes from the completeness of the ambient space.

We need one definition. Let (X, d) be a metric space. A subset E of X is called **nowhere dense** if its closure does not contain any metric ball. Equivalently, E is nowhere dense if $X \setminus E$ is dense in X. Note that a set is nowhere dense if and only if its closure is nowhere dense. The following result is called **Baire theorem**.

Theorem 6.4.1. Let {Ek}∞ 1 be a sequence of nowhere dense subsets of (X, d) where (X, d) is complete.

Then X \ S Ek is dense in X.

In particular, this theorem asserts that it is impossible to express a complete metric space as a countable union of nowhere sets. In applications, we often use it in the following form: Suppose $X = \setminus \bigcup E_k$.Then at least the closure of one of the Ek's has non-empty interior. An equivalent formulation is also useful: The intersection of countably many open dense sets in a complete metric space is again a dense set (though not necessarily open.)

Lemma 6.4.2. *Let* $\{\bar{B}j\}$ *be a sequence of closed balls in the complete metric space X which satisfies* $\bar{B}j+1 \subset \bar{B}j$ *and diam* $\bar{B}j \to 0$. *Then* $\bigcap_{j=1}^{\infty} \bar{B}_j$ *consists of a single point.*

Proof. Pick xj from Bj to form a sequence $\{xj\}$. As the diameters of the balls tend to zero, $\{xj\}$ is a Cauchy sequence. By the completeness of X, $\{xj\}$ converges to some x^*. Clearly x^* belongs to all Bj and is unique.

Proof of Theorem 6.1.10. By replacing Ej by its closure if necessary, we may assume all Ej's are closed sets. Let B_0 be any ball. We want to show that B_0 T(X \ Sj Ej) $\neq \emptyset$.

As $E1$ is nowhere dense and closed, we can find a closed ball $\bar{B}_1 \subset B_0$ such that $\bar{B}_1 \cap E1 = \emptyset$ and its diameter $d_1 \leq d_0/2$, the diameter of $B0$. Next, as $E2$ is nowhere dense and closed, by the same reason there is a closed ball $\bar{B}_2 \subset B1$ such that $\bar{B}_2 \cap E2 = \emptyset$ and $d2 \leq d1/2$. Repeating this process, we obtain a sequence of closed balls $\bar{B}j$ satisfying (1) $\overline{B_{j+1}} \subset \bar{B}j$, (2) $dj \leq d_0/2^j$, and (c) $\bar{B}j$ is disjoint from $E1, \cdots, Ej$. By Lemma 6.3.2 there is a point x^* in the common intersection of all Ej's. As $x^* \in \bar{B}j$ for all j, $x^* \in B_0 \setminus \cup j\ Ej$.

Baire theorem has many interesting applications.

Notes

Proposition 6.3.3. Any basis of an infinite dimensional Banach space contains uncountably many vectors.

Proof. First we claim any finite dimensional subspace of an infinite dimensional normed space is nowhere dense. Let E be such a subspace. As it is finite dimensional, it is closed. (Why?)

Pick $x_0 \in X \backslash E$, $||x0|| = 1$ (such $x0$ exists because X is of infinite dimensional). For any $x \in E$ and $\varepsilon > 0$, the point $x_\varepsilon = x + \varepsilon x_0 \in X \setminus E$ and $||x - x_\varepsilon|| < \varepsilon$, so $E = E$ does not contain any ball.

Let B be a countable basis of X, $B = \{x\}_{k=1}^{\infty}$. By the definition of a basis,

$$X = \bigcup_{n=1}^{\infty} E_n, \; E_n = \langle x_1, \cdots, x_n \rangle.$$

But this is impossible according to Baire theorem!

CHECK YOUR PROGRESS

3. State and prove Bairre's Theorem

---

4. Enumerate - *Any basis of an infinite dimensional Banach space contains uncountably many vectors.*

6.5 LET'S SUM UP

We concluded that the intersection of countably many open dense sets in a complete metric space is again a dense set (though not necessarily open.) . In the study of the well-posedness of solutions of partial differential equations we encounter numerous linear operators. This provides opportunity to apply the soft method of functional analysis to partial differential equations. Very often it is crucial to find the most appropriate spaces for the differential operator or its inverse operator to act on.

6.6 KEYWORDS

Solvable - **solvable** may refer to: **Solvable** group, a group that can be constructed by compositions of abelian groups, or equivalently a group whose derived series reaches the trivial group in finitely many steps.

Perpendicular - **Perpendicular** means "at right angles". A line meeting another at a right angle, or 90° is said to be **perpendicular** to it.

Transpose - To **transpose** something do the oppsite operation on it when carrying it across the equal sign.

6.7 QUESTION FOR REVIEW

1. Prove that $B(X, Y)$ is a Banach space when Y is a Banach space.

2. Show that in a complete metric space (X, d), the intersection of countably many open, dense
subsets is still a dense set.

3. Let $T \in B(X, Y)$ where X is a Banach space and Y is normed. Suppose there exists $C > 0$ such that

$$\|Tx\| \geq C\|x\|, \ \forall x \in X.$$

Show that
(a) $R(T)$ is a complete subspace of Y, and
(b) $T \in B(X, R(T))$ is invertible.

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6.9 ANSWER TO CHECK YOUR PROGRESS

1. Provide explanation – 6.1

2. Provide explanation and representation – 6.1.7

3. Provide statement and proof -6.3.1

4. Provide proof – 6.3.3

UNIT 7: BOUNDED LINEAR OPERATOR I

STRUCTURE

7.0 OBJECTIVE

Understand the concept of Uniform Boundedness Principle

Comprehend Open Mapping Theorem

Enumerate the concept of Spectrum

7.1 INTRODUCTION

In functional analysis, a bounded linear operator is a linear transformation L between normed vector spaces X and Y for which the ratio of the norm of L(v) to that of v is bounded above by the same number, over all non-zero vectors v in X. In other words, there exists some such that for all v in X.

7.2 UNIFORM BOUNDEDNESS PRINCIPLE

The following **uniform boundedness principle** is also called Banach-

Steinways theorem as a tribute to its discoverers. Steinhaus was the teacher of Banach.

Theorem 7.2.1. Let $\mathcal{T}$ be a family of bounded linear operators from a Banach space X to a normed space Y . Suppose that $\mathcal{T}$ is pointwisely bounded in the sense that for all x, there exists a constant Cx such that $\|Tx\| \leq Cx$ for all $T \in \mathcal{T}$. Then we can find a constant M such that $\|T\| \leq M$, for all $T \in \mathcal{T}$.

Proof. Let $Ek = \{x \in X : \|Tx\| \leq k$, for all $T \in \mathcal{T}\}$. We observe that

$$X = \bigcup_{k=1}^{\infty} E_k.$$

This is simply because for any $x \in X$, $\|Tx\| \leq Cx$ by assumption. Hence $x \in E_k$ for all $k \geq Cx$. Clearly each Ek is closed. By Baire theorem there is some E_{k0} which contains a ball B. It follows from the lemma below that $\|T\| \leq M$, for all $T \in \mathcal{T}$.

Lemma 7.2.2. *Let $T \in L(X, Y)$ where X and Y are normed spaces. Suppose that $\|TB_\rho(x_0)\| \leq C$. Then $\|T\| \leq 2C/\rho^{-1}$.*

Proof. As $B\rho(0) = B\rho(x0) - x0$, by linearity, we have

$$\|TB_\rho(0)\| \leq \|TB_\rho(x_0)\| + \|Tx_0\| \leq C + \|Tx_0\|,$$

$$\|T\| = \sup \|TB_1(0)\| \leq \frac{C + \|Tx_0\|}{\rho} \leq \frac{2C}{\rho}.$$

Uniform boundedness principle does not hold when the completeness of X is removed.

An alternative formulation of this principle is sometimes quite useful. A vector x_0 is called a **resonance point** for a family of bounded linear operators $\mathcal{T}$ if $\sup_{T\in\mathcal{T}} \|Tx_0\| = \infty$.

Theorem 7.2.3. *Let T be a family of bounded linear operators from X to Y where X is a Banach space and Y is a normed space. Suppose that $\sup_{T\in\mathcal{T}} \|T\| = \infty$. Then the resonance points of T forms a dense set in X.*

Proof. Suppose resonance points are not dense. There exists a ball $B_\rho(x_0)$ on which T is pointwisely bounded, that's, for all $x \in B_\rho(x_0)$ $||Tx|| \leq Cx$, for all $T \in T$. For any $x \in X$, $z = \rho x/||x||+x_0 \in B_\rho(x_0)$

$$\|T(\rho\frac{x}{\|x\|}+x_0)\| = \|Tz\| \leq C_z, \quad \text{for all } T \in \mathscr{T}$$

Implies

$$\|Tx\| \leq \frac{(C_z + \|Tx_0\|)}{\rho}\|x\|, \quad \text{for all } T \in \mathscr{T}.$$

So T is point wisely bounded on the whole X. By Banach-Steinways theorem, $||T|| \leq M$ for all T . But this is impossible by assumption. Recall that for any Riemann integrable function f of period 2π, its Fourier series is given by

$$\frac{a_0}{2} + \sum_1^\infty (a_n \cos nx + b_n \sin nx),$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\cos nydy, \quad n \geq 0,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\sin nydy, \quad n \geq 1.$$

We list the following facts (see, for instance, Stein and Shakarchi "Fourier Analysis"):

(1) The n-th partial sum S_nf of the Fourier series

$$(S_nf)(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$$

has a closed form

$$(S_nf)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin((n+\frac{1}{2})(y-x))}{\sin\frac{y-x}{2}}f(y)dy.$$

Notes

(2) It is believed that the formula

$$f(x) = \frac{a_0}{2} + \sum_1^{\infty}(a_k \cos kx + b_k \sin kx)$$

Should hold for "sufficiently nice functions".

(3) Taking $f_0(x) = 1$ $(0 \le x \le \pi)$ and $f_0(x) = 0$ $(-\pi \le x < 0)$ and extend it periodically in $\mathbb{R}$. The
Fourier series of f_0 is

$$\frac{1}{2} + \sum_{k \text{ odd}} \frac{2}{k\pi} \sin kx.$$

We have $f_0(0) = 0$ but the value of the Fourier series at 0 is 1 2. This shows that "sufficiently nice functions" should exclude discontinuous ones.

(4) For any Lipschitz continuous, 2π-periodic function f, its Fourier series converges uniformly to f everywhere. For continuous 2π-periodic functions it took some time to produce an example, see [Stein-Shakarchi] for an explicit construction.

Here we present a soft proof of a stronger result. Denote by $C(S^1)$ the vector space of all continuous, 2π-periodic functions. It can be identified with $\{f \in C[-\pi, \pi] : f(-\pi) = f(\pi)\}$, which is a closed subspace of $C[-\pi, \pi]$ under the sup-norm.

Theorem 7.1.4 . The subset {f ∈ $C(S^1)$: The Fourier series of f diverges at 0} is dense in C(S1). In particular, f is not equal to its Fourier series at 0.

Proof. We note that each partial sum $f \mapsto S_n f$ may be regarded as a linear operator, so composing with the evaluation at 0, $\Lambda_n f = (S_n f)(0)$, or

$$\Lambda_n f = \frac{a_0}{2} + \sum_1^n a_k,$$

forms a bounded linear functional on $C(S^1)$.

From the closed form of the partial sums we have

$$\Lambda_n f = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin((n+\frac{1}{2})y)}{\sin\frac{y}{2}} f(y)dy.$$

The integral kernel $K(x) = \sin((n + 1/2)x)/\sin x/2$ is continuous provided we set $K(0) = 2n + 1$. Its operator norm is equal to

$$\|\Lambda_n\| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|\sin((n+\frac{1}{2})y)|}{|\sin\frac{y}{2}|}dy$$

by the lemma below. We claim that $\sup_n ||\Lambda n|| = \infty$. This is done by a direct computation:

$$\begin{aligned}
\|\Lambda_n\| &= \frac{1}{\pi}\int_0^{\pi} \frac{|\sin((n+\frac{1}{2})y)|}{\sin\frac{y}{2}}dy \\
&\geq \frac{2}{\pi}\int_0^{\pi} \frac{|\sin((n+\frac{1}{2})y)|}{y}dy \quad (\because 0 \leq \sin\theta \leq \theta) \\
&= \frac{2}{\pi}\int_0^{(n+\frac{1}{2})\pi} \frac{|\sin x|}{x}dx \\
&\geq \frac{2}{\pi}\sum_{j=1}^{n}\int_{(j-1)\pi}^{j\pi} \frac{|\sin x|}{x}dx \\
&> \frac{2}{\pi}\sum_{j=1}^{n}\frac{1}{j\pi}\int_{(j-1)\pi}^{j\pi} |\sin x|dx \quad (\because \frac{1}{x} > \frac{1}{j\pi} \text{ for } x \in [(j-1)\pi, j\pi]) \\
&= \frac{4}{\pi^2}\sum_{j=1}^{n}\frac{1}{j} \to \infty
\end{aligned}$$

as $n \to \infty$. By Theorem 4.17, resonance points are dense in $C(S1)$. However, resonance points are precisely those functions in $C(S1)$ whose Fourier series diverges at 0.

Lemma 7.2.5. *Let*

$$\Lambda f = \int_a^b f(x)g(x)dx$$

where $g \in C[a, b]$. *Then* $\Lambda \in C[a, b]0$ *with*

$$\|\Lambda\| = \int_a^b |g(x)|dx.$$

Proof. This lemma could be proved by Riesz representation theorem. However, a direct proof is preferred.
Clearly we have

$$|\Lambda f| \leq \int_a^b |g(x)|dx,$$

for all f, $||f||_\infty \leq 1$. We need to establish the reverse inequality.

First assume that g is a polynomial p. Let I_k and J_k be open subintervals on which p is positive and negative respectively. For each small $\varepsilon > 0$, let I'_k and J'_k be subintervals of I_k and J_k respectively so that the distance between I'_k (resp. J'_k) and the endpoints of I_k (resp. Jk) is equal to ε. Define a function $f\varepsilon \in C[a, b]$ by setting it to be 1 on I'_k, -1 on J'_k, 0 at endpoints of I_k and J_k and linear in between.

Then $f\varepsilon \in C[a, b]$ with $||f\varepsilon||_\infty = 1$.

$$|\Lambda f_\varepsilon| = \left| \int_a^b (f_\varepsilon(x) - \mathrm{sgn} p(x))p(x)dx + \int_a^b |p(x)|dx \right| \geq \int_a^b |p(x)|dx - C\varepsilon,$$

or some constant C independent of ε.

$$\|\Lambda\| \geq \int_a^b |g(x)|dx,$$

when g is the polynomial p. By an approximation argument, this inequality also holds for every continuous g.
We end this section with a discussion on soft and hard methods in analysis. Very often a theorem
can be proved by two methods of very different nature. As a first example, consider the existence of transcendental numbers. Recall that an algebra number is a number that is a root for some polynomial in rational coefficients and a number is a transcendental number if it is not algebraic. In history the first transcendental number was found by Liouville (1844) who proved that Pj 10$-j$! is transcendental. The transcendentality of e and π were established by Hermite (1873) and

Lindemann (1882) respectively. However, using Cantor's theory of cardinality, it is easily shown that all algebraic numbers form a countable set. Since R is uncountable, the set of all transcendental numbers is equal to R minus all algebra numbers and therefore is uncountable. The soft method shows there are infinitely many transcendental numbers, but it cannot pinpoint which one is transcendental. In the previous section we discussed Baire theorem. As an application of this theorem, in the exercise you are asked to show that all continuous,
nowhere differentiable functions are dense in $C[0, 1]$. In 1872, Weierstrass caused a sensation in math community by constructing such functiond explicitly. This class of functions are given by

$$\sum_j a^j \cos(b^n \pi x),$$

where $a \in (0, 1)$, b an odd integer, $ab > 1 + 3\pi/2$. Again the soft method cannot give you any explicit example.
Finally, in the above discussion we proved that the collection of all periodic, continuous functions whose Fourier series are divergent at 0 is a dense subset of $C(S1)$, but again we cannot tell which one belongs to this collection. You need to find it in a hard way.

7.3 OPEN MAPPING THEOREM

The open mapping theorem asserts that a subjective bounded linear operator from a Banach space to another Banach space must be an open map. This result is uninteresting in the finite dimensional situation, but turns out to be very important for infinite dimensional spaces. From history there were several concrete, relevant results in various areas, Banach had the insight to single out the property as a theorem.

A map $f : (X, d) \mapsto (Y, \rho)$ between two metric spaces is called an **open map** if $f(G)$ is open in Y for any open set G in X. This should not be confused with continuity of a map, namely, f is continuous if $f^{-1}(E)$ is open in X for any open set E in Y. As an example, let us show that every non-zero linear functional on a normed space X is an open map. Indeed, pick $z_0 \in X$ with $\Lambda_{z0} = 1$. Such point always exists when the functional Λ is non-zero. For any open set G in X, we claim that ΛG is open. Letting $\Lambda_{x0} \in \Lambda G$, as $x_0 \in G$ and G is open, there exists some $R > 0$ such that $BR(x0)$ is contained in G. Then $x_0 + r_{z0} \in BR(x_0)$ for all $r \in (-R, R)$ and $\Lambda(x_0 + r_{z0}) = \Lambda x_0 + r$ imply that $(\Lambda x0 + R, \Lambda x0 - R) \in \Lambda G$, so ΛG is open.

Before stating the theorem, let's state a necessary and sufficient condition for a linear operator to be open.

Lemma 7.2.1. Let T ∈ L(X, Y) when X and Y are normed spaces. T is an open map if the image of a ball under T contains a ball.

Roughly speaking, a linear operator either has "fat" image or it collapses everywhere.

Proof. We use "D" instead of "B" to denote a ball in Y. Suppose there exists D_{r_0} $(Tx_1) \subset TBR_0(x_0)$ for some $x1 \in BR_0(x_0)$. By linearity,

$$D_{r_0}(Tx_1) = D_{r_0}(0) + Tx_1 \subset TB_{R_0}(x_0) \text{ implies}$$

$$\begin{aligned} D_{r_0}(0) &\subset TB_{R_0}(x_0) - Tx_1 \\ &= TB_{R_0}(x_0 - x_1) \\ &\subset TB_{R_1}(0), \quad R_1 = R_0 + \|x_0 - x_1\|. \end{aligned}$$

Let G be an open set in X. We want to show that TG is open. So, for $Tx_0 \in TG$, $x0 \in G$, as G is open, we can find a small $\rho > 0$ such that $B\rho(x0) \subset G$. From the above inclusion,

$$D_\varepsilon(0) \subset TB_\rho(0), \qquad \varepsilon = \rho\frac{r_0}{R_1},$$

$$D_\varepsilon(Tx_0) \subset TB_\rho(x_0)$$

Theorem 7.2.2. *Any surjective bounded linear operator from a Banach space to another Banach space is an open map.*

Unlike the uniform boundedness principle here we require both the domain and target of the linear operator be complete.

Proof. Step 1: We claim that there exists $r > 0$ such that

$$D_r(0) \subset \overline{TB_1(0)}.$$

For, as T is onto, we have

$$Y = \bigcup_{1}^{\infty} TB_j(0) = \bigcup_{1}^{\infty} \overline{TB_j(0)}.$$

By assumption Y is complete, so we may apply Baire theorem to conclude that TB_{j_0} (0) contains a ball for some j_0, i.e.,

$$D_\rho(y_0) \subset \overline{TB_{j_0}(0)}.$$

Since TB_{j_0} (0) is dense in TB_{j_0} (0), by replacing $D\rho(y0)$ by a smaller ball if necessary, we may assume $y_0 = Tx_0$, for some $x0 \in B_{j_0}$ (0). Then

$$D_\rho(y_0) \subset \overline{TB_{j_0}(0)} \subset \overline{TB_R(x_0)},\ \ R = j_0 + \|x_0\|,$$

$$D_\rho(0) \subset \overline{TB_R(0)},$$

$$D_r(0) \subset \overline{TB_1(0)}, \qquad r = \frac{\rho}{R}.$$

Step 2: $Dr(0) \subset TB_3(0)$.

First, note by scaling,

$$D_{\frac{r}{2^n}}(0) \subset \overline{TB_{\frac{1}{2^n}}(0)}, \quad \text{for all } n \geq 0 \qquad (1)$$

Letting $y \in Dr(0)$, we want to find $x* \in B3(0)$, $T x* = y$. We will do this by constructing an approximating sequence.

For $\varepsilon = r/2$, from (1) with $n = 0$, there exists $x1 \in B1(0)$ such that

$$\|y - Tx_1\| < \frac{r}{2}.$$

As $y - T x1 \in D_{r/2}(0)$, for $\varepsilon = \frac{r}{2^2}$, from (1) with $n = 1$, there exists $x_2 \in B_{1/2}(0)$ such that

$$\|y - Tx_1 - Tx_2\| < \frac{r}{2^2}.$$

Keep doing this we get $\{xn\}$,

$$x_n \in B_{\frac{1}{2^{n-1}}}(0)$$

such that

$$\|y - Tx_1 - Tx_2 - \cdots - Tx_n\| < \frac{r}{2^n}.$$

Setting $z_n = \sum_1^n x_j$, we have

$$\|y - Tz_n\| < \frac{r}{2^n}.$$

Let's verify that *{zn}* is a Cauchy sequence in *X*. ∀*n*, *m*, *m* < *n*,

$$\begin{aligned} \|z_n - z_m\| &= \|x_{m+1} + \cdots + x_n\| \\ &\leq \|x_{m+1}\| + \cdots + \|x_n\| \\ &< \frac{1}{2^m} + \cdots + \frac{1}{2^n} \leq \frac{1}{2^{m-1}} \to 0 \end{aligned}$$

as $m \longrightarrow \infty$. From the completeness of *X* we may set $z* = \lim_{n\to\infty} z_n$. Let's check that $z* \in B3(0)$ and $T z* = y$. For,

$$\|z_n\| \leq \sum_1^n \|x_j\| \leq \sum_1^n \frac{1}{2^{j-1}} \leq 2 < 3.$$

So $z*$ belongs to the closure of $B_2(0)$, or, in $B_3(0)$. Next,

$$\begin{aligned} \|y - Tz^*\| &\leq \|y - Tz_n\| + \|Tz_n - Tz^*\| \\ &\leq \frac{r}{2^n} + \|T\|\|z_n - z^*\| \to 0 \end{aligned}$$

as $n \longrightarrow \infty$, so $y = T z$.

We have shown that the image of the ball $B_3(0)$ under *T* contains the ball *Dr*(0), and the desired conclusion follows from Lemma 7.1.3. Recall that a linear operator is invertible if it is bounded, bijective and with a bounded inverse. The following theorem shows that the boundedness of the inverse comes as a consequence of boundedness and surjectivity of the operator when working on Banach spaces. This is called the Banach inverse mapping theorem.

Corollary 7 .3.3. *Let T ∈ B(X, Y) be a bijection where X and Y are Banach spaces. Then T is invertible.*

Proof. It suffices to show that the inverse map T^{-1} is bounded. From the above proof $D_r(0) \subset T B_3(0)$ holds. As T is bijective, $T^{-1}(D_r(0)) \subset B_3(0)$. In other words, T^{-1} maps a ball in Y to a bounded set in X, so T^{-1} is bounded.

A theorem in general topology asserts that a continuous bijection from $\mathbb{R}^n$ to $\mathbb{R}^n$ must have a continuous inverse, that is, it is a homeomorphism. This property does not hold for continuous maps in a general Banach space. However, it remains to be valid when the map is a bounded linear operator.

A standard application of the open mapping theorem is the closed graph theorem. By definition a

linear operator T between normed spaces X and Y is called a **closed map** or **of closed graph** if the graph of T, $G(T) \equiv \{(x, Tx) : x \in X\} \subset X \times Y$, is a closed set in the product normed space $X \times Y$. Observe that $X \times Y$ is also a subspace of $X \times Y$.

An alternative definition is, T is closed if whenever $x_n \to x$ and $Tx_n \to y$, we have $y = Tx$. From the definition one sees immediately that any bounded linear operator is a closed map. But the converse is not always true. As an exercise you may check that the differential operator is a closed map; but we already showed that it is unbounded. The following **closed graph theorem** provides an efficient way to verify the boundedness of a linear operator.

Theorem 7.3.4 . Any closed map from a Banach space to another Banach space is bounded.

Proof. Let T be a closed map in $L(X, Y)$ where X and Y are Banach spaces. Since $X \times Y$ is complete and $G(T)$ is closed in $X \times Y$ by assumption, $G(T)$ is also a Banach space. We consider the linear map P which is simply the projection of $G(T)$ to X: $P(x, Tx) = x$. Clearly P is bijective. From the relation

$$\|P(x, Tx)\| = \|x\| \leq \|x\| + \|Tx\|,$$

we see that P belongs to $B(G(T), X)$. By the above corollary we conclude that P^{-1} is bounded. There exists some constant C such that

$$\|x\| + \|Tx\| = \|P^{-1}x\| \leq C\|x\|.$$

In particular, we have $||Tx|| \leq C||x||$.
With more effort, one can deduce the open mapping theorem from the closed graph theorem. So these two results are in fact equivalent.

CHECK YOUR PROGRESS

1. State Uniform Boundness Principle

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2. State and prove **Banach inverse mapping theorem**

7.4 THE SPECTRUM

Denote by $B(X) = B(X, X)$ the vector space of all bounded linear operators from the normed space X to itself. It is a normed space under the operator norm, and it is a Banach space when X is a Banach space. An additional algebraic operation, namely, the composition of two linear operators, makes sense in $B(X)$. In fact, we note that

(i) the identity map I is well-defined in $B(X)$;
(ii) for all $T, S \in B(X)$, $TS \in B(X)$ and $||TS|| \leq ||T||||S||$.

$B(X)$ is the prototype for Banach algebras. When the space is of finite dimension, one may regard it as $\mathbb{F}^n$, so any linear operator is essentially a square matrix. In the theory of square matrices eigenvalues and eigenvectors are of central importance. In general, it is possible to define the same notion for linear operators in $B(X)$. A scalar λ is called an **eigenvalue** for T in $L(X, X)$ if there exists a nonzero x, called an **eigenvector**, such that

$$Tx = \lambda x.$$

As before, it is readily checked that all eigenvectors form a subspace together with 0. It is closed when X is normed and T is bounded.
In the following we let X be a Banach space and the linear operator T bounded for simplicity, although much of the discussion could be extended to more general settings. Recall that a bounded linear operator S in $B(X)$ is called invertible if it is bijective and $S-1$ belongs to $B(X)$. According to the open mapping theorem, S is invertible if and only if it is bijective.
A scalar $\lambda \in \mathbb{F}$ is called a **regular value** for a bounded linear operator T if $T - \lambda I$ is invertible. The set of all regular values of T forms the **resolvent set** of T, denoted by $\rho(T)$, and we define its complement, that's, $\mathbb{F} \setminus \rho(T)$, the **spectrum** of T and denote it by $\sigma(T)$. Both terminologies are motivated by their connection with physics.
We point out that any eigenvalue λ of the bounded linear operator T must belong to the spectrum of T. Indeed, the existence of an eigenvector shows that $T - \lambda I$ is not injective, hence cannot be invertible. When the space is finite dimensional, a linear operator is injective if and only if it is surjective. Consequently, the spectrum of any linear operator consists exactly of eigenvalues. However, this is no longer the case for infinite dimensional spaces. For T on a Banach space, $T - \lambda I$ fails to be invertible for two reasons; either it is not injective or not surjective. The scalar λ is an eigenvalue when the former holds. An example may be helpful in illustrating the situation.

Let $X = C[0, 1]$ over the real field and consider the linear operator T given by $(Tf)(x) = xf(x)$.
Clearly, $T \in B(C[0, 1])$ with $||T|| \leq 1$. If λ is an eigenvalue of T, and ϕ its eigenfunction, $x\phi(x) = \lambda\phi(x)$ will hold for all $x \in [0, 1]$. This is clearly impossible, so T does not have any eigenvalue. However, for any λ not in $[0, 1]$, the inverse of $T - \lambda I$ is given by the map

$$(Sf)(x) = \frac{f(x)}{x - \lambda}.$$

It is easy to check that $S \in B(C[0, 1])$. When $\lambda \in [0, 1]$, the inverse of $T - \lambda I$ does not exist. We conclude that although T has no eigenvalues, its spectrum is given by $\sigma(T) = [0, 1]$ and resolvent set by C/[0, 1].

Proposition 7.2.1 . *Let $T \in B(X)$ where X is a Banach space. Then $I - T$ is invertible when $||T|| < 1$.*

Corollary 7.2.2 *The spectrum of $T \in B(X)$ where X is a Banach space forms a closed and bounded set in* F. *In fact, $|\lambda| \leq ||T||$ for any $\lambda \in \sigma(T)$.*

Proof. If λ does not belong to $\sigma(T)$, that is, $T - \lambda I$ is invertible. There exists $\rho > 0$ such that all linear operators in $B\rho(T - \lambda I)$ are invertible. In particular, it means $T - \mu I$, $|\lambda - \mu| < \rho$, is invertible. This shows that the complement of $\sigma(T)$ is open, hence $\sigma(T)$ is closed. Next, if $|\lambda| > ||T||$, then $I - \lambda^{-1} T$ and hence $T - \lambda I$ are invertible by Proposition 4.21. Hence λ cannot be in the spectrum.

Evidently there is a natural question: Is the spectrum nonempty for any bounded linear operator

in $B(X)$? After all, there are n many eigenvalues (including multiplicity) for any $n \times n$- matrix with complex entries. Remember that the proof of this fact depends on the fundamental theorem of algebra which is most easily established by using the Liouville theorem in complex analysis. It is not surprising we need to use complex analysis to establish the following two results over C:

First, $\sigma(T)$ is always nonempty for any $T \in B(X)$;

Second, we have the formula for the "spectral radius":

$$\sup\{|\lambda| : \lambda \in \sigma(T)\} = \lim_{k\to\infty} \|T^k\|^{1/k}.$$

Theorem 7.2.2. *Let $T \in B(X)$ where X is a complex Banach space. Then*

(a) $\rho(T)$ is open in C. *More precisely, for any $\lambda 0 \in \rho(T)$, $\lambda \in \rho(T)$ for $|\lambda - \lambda 0| < 1/||(\lambda_0 I - T)^{-1}||$.*

(b) For each $\Lambda \in B(X)'$, the function $\phi(\lambda) = \Lambda(\lambda I - T)^{-1}$ is analytic in $\rho(T)$.

(c) $\sigma(T)$ is a non-empty compact set in the plane.

Proof. (a) Follows immediately from Theorem 4.4 after taking $T = \lambda 0I -$ T and $S = \lambda I - T$ in that theorem.

(b). To show analyticity we represent $\phi(\lambda)$ as a power series around every $\lambda 0$ in $\rho(T)$. Formally, we have $\lambda I - T = (\lambda 0I - T)[1 + (\lambda - \lambda 0)(\lambda 0I - T$ $)-1]$, so define

$$(\lambda I - T)^{-1} = (\lambda_0 I - T)^{-1} \sum_{k=0}^{\infty} (-1)^k (\lambda_0 I - T)^{-k} (\lambda - \lambda_0)^k.$$

When $|\lambda - \lambda 0| < 1/k(\lambda 0I - T)-1k$, there exists some $\sigma \in (0, 1)$ such that $k(\lambda 0I - T)-1k|\lambda - \lambda 0| < 1 - \sigma$, therefore, this power series converges and one can easily check that it converges to $(\lambda I - T)-1$. Hence the above formal expression holds rigorous. For $\Lambda \in B(X)0$, we have

$$\varphi(\lambda) = \sum_{k=0}^{\infty} (-1)^k \Lambda((\lambda_0 I - T)^{-k-1})(\lambda - \lambda_0)^k$$

also converges for λ, $|\lambda - \lambda_0| < 1/k(\lambda_0 I - T)-1k$.

(c). We first show that $\phi(\lambda) \to 0$ as $|\lambda| \to \infty$ for any $\Lambda \in B(X)0$. In this time we expand ϕ at ∞. Formally

$$(\lambda I - T)^{-1} = \frac{1}{\lambda}\left(I - \frac{T}{\lambda}\right)^{-1} = \frac{1}{\lambda}\sum_{k=0}^{\infty}\frac{T^k}{\lambda^k}.$$

For $\lambda > ||T||$, this can be made rigorously and so

$$\begin{aligned} |\varphi(\lambda)| &= |\Lambda(\lambda I - T)^{-1}| \\ &\leq \frac{1}{|\lambda|}\sum_{k=0}^{\infty}\frac{|\Lambda T^k|}{|\lambda^k|} \\ &\leq \frac{C}{|\lambda|}||\Lambda|| \to 0 \end{aligned}$$

as $|\lambda| \to \infty$.

If $\sigma(T)$ is empty, that means ϕ is an entire function. As it tends to 0 at ∞, it is bounded on C. By Lowville theorem we conclude that ϕ is identically zero for every $\Lambda \in B(X)0$. By Hahn-Banach theorem, $\lambda I - T =$ 0 for all λ, contradiction holds. Hence the spectrum is always non-empty.

Notes

Define the **spectral radius** of $T \in B(X)$ by

$$rT = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

We know that $0 \leq rT \leq kTk$. We have a precise formula.

Theorem 7.3.3. *For any $T \in B(X)$ where X is a Banach space,*

$$r_T = \overline{\lim_{n\to\infty}} \sqrt[n]{\|T^n\|}.$$

CHECK YOUR PROGRESS

3. Define Closed Map

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4. State **closed graph theorem**

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5. State the formula for the "spectral radius"

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7.5 LET'S SUM UP

Unbounded operators play an important role in quantum physics. Unfold relationship and how Banach space and the linear operator T bounded. The open mapping theorem asserts that a surjective bounded linear operator from a Banach space to another Banach space must be an open map. This result is uninteresting in the finite dimensional situation, but turns out to be very important for infinite dimensional spaces.

7.6 KEYWORDS

Resonance points- The phenomenon of increasing amplitudes of forced oscillations when the frequency of the external action approximates one of the frequencies of the eigenoscillations (cf. Eigen oscillation) of a dynamical system

Scalar - A quantity all values of which can be expressed by one (real) number. More generally, a scalar is an element of some field.

Regular value- A scalar $\lambda \in$ F is called a **regular value** for a bounded linear operator T if $T - \lambda I$ is invertible.

Resolvent set - The set of all regular values of T forms the **resolvent set** of T, denoted by $\rho(T)$

7.7 QUESTION FOR REVIEW

1. Prove : - *For any* $T \in B(X)$ *where* X *is a Banach space*

$$r_T = \overline{\lim_{n\to\infty}} \sqrt[n]{\|T^n\|}.$$

2. Deduce the open mapping theorem from the closed graph theorem. Suggestion: Consider $X/N(T)$ and the closed map T e given by $\tilde{T}(\tilde{x}) = Tx$ on $X/N(T)$.

3. Let $T \in B(X)$ where X is normed. Show that $\sigma(T) = \sigma(T\,0)$. Hint: Show that S is invertible if and only if S' is invertible.

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7.9 ANSWER TO CHECK YOUR PROGRESS

1. Provide statement– 7.1.1

2. Provide statements and proof – 7.2.3

3. Provide statement below corollary -- 7.2.3

4. Provide statement and proof -7.2.4

5. Refer formula – 7.3.2